

# EXISTENCE OF HILBERT CUSP FORMS WITH NON-VANISHING $L$ -VALUES

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**ABSTRACT.** We give a derivative version of the relative trace formula on  $\mathrm{PGL}(2)$  studied in our previous work, and obtain a formula of an average of central values (derivatives) of automorphic  $L$ -functions for Hilbert cusp forms. As an application, we prove existence of Hilbert cusp forms with non-vanishing central values (derivatives) such that the absolute degrees of their Hecke fields are sufficiently large.

## 1. INTRODUCTION

This is a continuation of our previous paper [7]; we freely use the notation introduced there, which is collected at the end of this section for a convenience of reference.

Let  $F$  be a totally real number field of degree  $d_F$  and  $\mathfrak{o}$  the integer ring of  $F$ . Let  $\mathfrak{n}$  be an  $\mathfrak{o}$ -ideal and  $l = (l_v)_{v \in \Sigma_\infty} \in (2\mathbb{N})^{d_F}$  an even weight. For  $\pi \in \Pi_{\mathrm{cus}}^*(l, \mathfrak{n})$  and an idele class character  $\eta$  of  $F^\times$  such that  $\eta^2 = \mathbf{1}$ , the standard  $L$ -function  $L(s, \pi \otimes \eta)$  of  $\pi \otimes \eta$  is an entire function on  $\mathbb{C}$  satisfying the functional equation

$$L(s, \pi \otimes \eta) = \epsilon(s, \pi \otimes \eta) L(1-s, \pi \otimes \eta),$$

with  $\epsilon(s, \pi \otimes \eta)$  being the  $\epsilon$ -factor; it is of the form  $\epsilon(s, \pi \otimes \eta) = \pm (N(\mathfrak{n} \mathfrak{f}_\eta^2) D_F^2)^{1/2-s}$ . The number  $\epsilon(1/2, \pi \otimes \eta) \in \{+1, -1\}$ , is called the sign of the functional equation. The central value  $L(1/2, \pi) L(1/2, \pi \otimes \eta)$  and the derivative  $L(1/2, \pi) L'(1/2, \pi \otimes \eta)$  has an important arithmetic meaning; there are many works which exploit the nature of these  $L$ -values in connection with the arithmetic algebraic geometry of modular varieties ([12], [1], [13], [14], [15], [16]).

1.1. Let  $\mathfrak{a} \subset \mathfrak{o}$  be an ideal relatively prime to  $\mathfrak{f}_\eta \mathfrak{n}$  and set  $S = S(\mathfrak{a})$ . We write the Satake parameter of  $\pi \in \Pi_{\mathrm{cus}}^*(l, \mathfrak{n})$  at  $v \in S$  as  $\mathrm{diag}(q_v^{\nu_v(\pi)/2}, q_v^{-\nu_v(\pi)/2})$  with  $\pm \nu_v(\pi)$  belonging to the space  $\mathfrak{X}_v = \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$ . In [7], given an even holomorphic function  $\alpha(\mathbf{s})$  on  $\mathfrak{X}_S = \prod_{v \in S} \mathfrak{X}_v$ , we studied the asymptotic of the average

$$(1.1) \quad \mathrm{AL}^*(\mathfrak{n}, \alpha) = \frac{C_l}{N(\mathfrak{n})} \sum_{\pi \in \Pi_{\mathrm{cus}}^*(l, \mathfrak{n})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \mathrm{Ad})} \alpha(\nu_S(\pi))$$

with  $\nu_S(\pi) = \{\nu_v(\pi)\}_{v \in S}$  and

$$(1.2) \quad C_l = \prod_{v \in \Sigma_\infty} \frac{2\pi (l_v - 2)!}{\{(l_v/2 - 1)!\}^2}$$

as the norm  $N(\mathfrak{n})$  grows under the following conditions.

- (a) The number  $(-1)^{\epsilon(\eta)} \tilde{\eta}(\mathfrak{n})$ , the common value of  $\epsilon(s, \pi) \epsilon(s, \pi \otimes \eta)|_{s=1/2}$  for all  $\pi \in \Pi_{\mathrm{cus}}^*(l, \mathfrak{n})$ , equals 1.

(b)  $\eta_v(\varpi_v) = -1$  for any  $v \in S(\mathbf{n})$ .

In this paper, imposing the same condition (b) as above but the different sign condition  $(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathbf{n}) = -1$  than (a), we investigate the asymptotic behavior of the following average involving the central derivative of  $L$ -function  $L(s, \pi \otimes \eta)$ .

$$(1.3) \quad \text{ADL}_-^*(\mathbf{n}, \alpha) = \frac{C_l}{N(\mathbf{n})} \sum_{\substack{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}) \\ \epsilon(1/2, \pi \otimes \eta) = -1}} \frac{L(1/2, \pi) L'(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \text{Ad})} \alpha(\nu_S(\pi)).$$

To state our main result precisely, we need notation. Let  $\mathcal{I}_{S, \eta}$  be the monoid of ideals  $\mathbf{n} \subset \mathfrak{o}$  generated by prime ideals  $\mathfrak{p}$  prime to  $S \cup S(\mathfrak{f}_\eta)$  such that  $\tilde{\eta}(\mathfrak{p}) = -1$ , and

$$\mathcal{I}_{S, \eta}^\pm = \{\mathbf{n} \in \mathcal{I}_{S, \eta} \mid (-1)^{\epsilon(\eta)}\tilde{\eta}(\mathbf{n}) = \pm 1\}.$$

For  $n \in \mathbb{N}$ , let  $X_n(x)$  be the Tchebyshev polynomial  $X_n(x)$  defined by the relation

$$(1.4) \quad X_n(x) = \sin((n+1)\theta)/\sin \theta \quad \text{for } x = 2 \cos \theta$$

and set

$$(1.5) \quad \alpha_{\mathbf{a}}(\nu) = \prod_{v \in S} X_{n_v}(q_v^{\nu_v/2} + q_v^{-\nu_v/2}), \quad \nu = \{\nu_v\}_{v \in S} \in \mathfrak{X}_S$$

in terms of the prime ideal decomposition  $\mathbf{a} = \prod_{v \in S} \mathfrak{p}_v^{n_v}$ . For such  $\mathbf{a}$ , define  $\mathbf{a}_\eta^\pm = \prod_{\substack{v \in S(\mathbf{a}) \\ \tilde{\eta}(\mathfrak{p}_v) = \pm 1}} \mathfrak{p}_v^{n_v}$ ,  $d_1(\mathbf{a}) = \prod_{v \in S(\mathbf{a})} (n_v + 1)$  and  $\delta_\square(\mathbf{a}) = \prod_{v \in S(\mathbf{a})} \delta(n_v \in 2\mathbb{N})$ . We have an asymptotic formula of  $\text{ADL}_-^*(\mathbf{n}; \alpha_{\mathbf{a}})$  with an error term whose dependence on  $\mathbf{n} \in \mathcal{I}_{S, \eta}^-$  and  $\mathbf{a}$  is made explicated. We also have a similar formula for  $\text{AL}^*(\mathbf{n}; \alpha_{\mathbf{a}})$  with  $\mathbf{n} \in \mathcal{I}_{S, \eta}^+$ .

**Theorem 1.** *Suppose  $\underline{l} = \min_{v \in \Sigma_\infty} l_v \geq 6$ . Set  $c = d_F^{-1}(\underline{l}/2 - 1)$ . For an integral ideal  $\mathbf{n}$ , set*

$$\nu(\mathbf{n}) = \left\{ \prod_{v \in S(\mathbf{n}) - (S_1(\mathbf{n}) \cup S_2(\mathbf{n}))} (1 - q_v^{-2}) \right\} \left\{ \prod_{v \in S_2(\mathbf{n})} (1 - (q_v^2 - q_v)^{-1}) \right\}.$$

For any sufficiently small number  $\epsilon > 0$ , we have

$$(1.6) \quad \begin{aligned} \text{AL}^*(\mathbf{n}; \alpha_{\mathbf{a}}) &= 4D_F^{3/2} L_{\text{fin}}(1, \eta) \nu(\mathbf{n}) N(\mathbf{a})^{-1/2} \delta_\square(\mathbf{a}_\eta^-) d_1(\mathbf{a}_\eta^+) \\ &\quad + \mathcal{O}_{\epsilon, l, \eta} \left( N(\mathbf{a})^{c+2+\epsilon} N(\mathbf{n})^{-\inf(c, 1)+\epsilon} \right), \quad \mathbf{n} \in \mathcal{I}_{S, \eta}^+, \end{aligned}$$

$$(1.7) \quad \begin{aligned} &\text{ADL}_-^*(\mathbf{n}; \alpha_{\mathbf{a}}) \\ &= 4D_F^{3/2} L_{\text{fin}}(1, \eta) \nu(\mathbf{n}) N(\mathbf{a})^{-1/2} d_1(\mathbf{a}_\eta^+) \left\{ \delta_\square(\mathbf{a}_\eta^-) \left( \log(\sqrt{N(\mathbf{n})N(\mathbf{a})^{-1}}N(\mathfrak{f}_\eta)D_F) \right. \right. \\ &\quad \left. \left. + \sum_{v \in S(\mathbf{n}) - (S_1(\mathbf{n}) \cup S_2(\mathbf{n}))} \frac{\log q_v}{q_v^2 - 1} + \sum_{v \in S_2(\mathbf{n})} \frac{\log q_v}{q_v^2 - q_v - 1} + \frac{L'}{L}(1, \eta) + \mathfrak{C}(l) \right) \right. \\ &\quad \left. + \sum_{v \in S(\mathbf{a}_\eta^-)} \delta_\square(\mathbf{a}_\eta^- \mathfrak{p}_v^{-1}) \log(q_v^{n_v + \frac{1}{2}}) \right. \\ &\quad \left. + \mathcal{O}_{\epsilon, l, \eta} \left( N(\mathbf{a})^{-1/2} d_1(\mathbf{a}_\eta^+) \delta_\square(\mathbf{a}_\eta^-) X(\mathbf{n}) + N(\mathbf{a})^{c+2+\epsilon} N(\mathbf{n})^{-\inf(1, c)+\epsilon} \right) \right\}, \quad \mathbf{n} \in \mathcal{I}_{S, \eta}^-, \end{aligned}$$

where

$$\mathfrak{C}(l) = \sum_{v \in \Sigma_\infty} \left( \sum_{k=1}^{l_v/2-1} \frac{1}{k} - \frac{1}{2} \log \pi - \frac{1}{2} C_{\text{Euler}} - \delta(\eta_v(-1) = -1) \log 2 \right),$$

$$X(\mathfrak{n}) = \sum_{u \in S(\mathfrak{n})} \frac{\log q_u}{q_u} + \sum_{u \in S(\mathfrak{n})} \frac{\log q_u}{(q_u - 1)^2}.$$

The constants implicit in the  $O$ -symbols in both formulas are independent of  $\mathfrak{n}$  and  $\mathfrak{a}$ .

1.2. For a positive integer  $N$ , let  $J_0^{\text{new}}(N)$  be the new part of the Jacobian variety of the modular curve  $X_0(N)$  of level  $N$ . J.-P. Serre showed that the largest dimension of  $\mathbb{Q}$ -simple factors of  $J_0^{\text{new}}(N)$  tends to infinity as  $N$  grows ([11, Theorem 7]). This result was refined in several ways by E. Royer ([4]); he obtained a quantitative version of Serre's theorem giving a lower bound of the largest dimension of  $\mathbb{Q}$ -simple factors  $A$  of  $J_0^{\text{new}}(N)$  with or without rank conditions for the Model-Weil group of  $A$ . By the correspondence between the  $\mathbb{Q}$ -simple factors  $A$  of  $J_0^{\text{new}}(N)$  and the normalized Hecke eigen newforms  $f$  of level  $\Gamma_0(N)$  and weight 2, and by invoking the progress toward the Birch and Swinnerton-Dyer conjecture, the lower bound for the largest  $\dim A$  is obtained from a lower bound of the maximum value of the absolute degree of the Hecke field  $\mathbb{Q}(f)$  with or without conditions on the order of  $L$ -series  $L(s, f)$  at the center of symmetry. Thus, one of Royer's results in [4] can be stated in the language of modular forms as follows.

**Theorem 2.** (Royer [4]) *Let  $p$  be a prime. There exist constants  $C_p > 0$  and  $N_p > 0$  with the following properties:*

- (1) *For any  $N > N_p$  relatively prime to  $p$ , there exists a normalized Hecke eigen newform  $f$  of level  $\Gamma_0(N)$  and weight 2 satisfying the conditions:*
  - (i)  *$L(1/2, f) \neq 0$ , where the functional equation of  $L(s, f)$  relates the values at  $s$  and  $1 - s$ .*
  - (ii)  *$[\mathbb{Q}(f) : \mathbb{Q}] \geq C_p \sqrt{\log \log N}$ .*
- (2) *For any  $N > N_p$  relatively prime to  $p$ , there exists a normalized Hecke eigen newform  $f_1$  of level  $\Gamma_0(N)$  and weight 2 satisfying the conditions:*
  - (i) *The sign of the functional equation of  $L(s, f_1)$  is  $-1$ .*
  - (ii)  *$L'(1/2, f_1) \neq 0$ .*
  - (iii)  *$[\mathbb{Q}(f_1) : \mathbb{Q}] \geq C_p \sqrt{\log \log N}$ .*

We obtain an analogue of this theorem for higher weight Hilbert modular cuspforms by using Theorem 1. For a cupsidal representation  $\pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n})$ , we denote by  $\mathbb{Q}(\pi)$  the field of rationality of  $\pi$  (for definition, see 7.1.)

**Theorem 3.** *Let  $l = (l_v)_{v \in \Sigma_\infty}$  be a weight such that  $l_v = k$  for all  $v \in \Sigma_\infty$  with an even integer  $k \geq 6$ , and  $\eta$  a quadratic idele class character of  $F^\times$ . Let  $S$  be a finite subset of  $\Sigma_{\text{fin}} - S(\mathfrak{f}_\eta)$  and  $\mathbf{J} = \{J_v\}_{v \in S}$  a family of closed subintervals of  $(-2, 2)$ . Given a prime ideal  $\mathfrak{q}$  prime to  $S \cup S(\mathfrak{f}_\eta)$ , there exist constants  $C_{\mathfrak{q}} > 0$  and  $N_{\mathfrak{q}, S, l, \eta, \mathbf{J}} > 0$  with the following properties: For any ideal  $\mathfrak{n} \in \mathcal{I}_{S \cup S(\mathfrak{q}), \eta}^+$  with  $N(\mathfrak{n}) > N_{\mathfrak{q}, S, l, \eta, \mathbf{J}}$ , there exists  $\pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n})$  such that*

- (i)  *$L(1/2, \pi) \neq 0$  and  $L(1/2, \pi \otimes \eta) \neq 0$ ,*
- (ii)  *$[\mathbb{Q}(\pi) : \mathbb{Q}] \geq C_{\mathfrak{q}} \sqrt{\log \log N(\mathfrak{n})}$ , and*

(iii)  $q_v^{\nu_v(\pi)/2} + q_v^{-\nu_v(\pi)/2} \in J_v$  for all  $v \in S$ .

We should note that this can be regarded as a refinement of [7, Corollary 1.2].

As for derivatives, we have a conditional result.

**Theorem 4.** *Let  $l = (l_v)_{v \in \Sigma_\infty}$  and  $\eta$  be the same as in Theorem 3. Suppose that for any ideal  $\mathfrak{n}$  prime to  $\mathfrak{f}_\eta$ ,*

$$(1.8) \quad \frac{d}{ds} \Big|_{s=1/2} (L(s, \pi) L(s, \pi \otimes \eta)) \geq 0 \quad \text{for all } \pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n}).$$

*Let  $S$  be a finite subset of  $\Sigma_{\text{fin}} - S(\mathfrak{f}_\eta)$  and  $\mathbf{J} = \{J_v\}_{v \in S}$  a family of closed subintervals of  $(-2, 2)$ . Given a prime ideal  $\mathfrak{q}$  prime to  $S \cup S(\mathfrak{f}_\eta)$  and a constant  $M > 1$ , there exist constants  $C_{\mathfrak{q}} > 0$  and  $N_{\mathfrak{q}, S, l, \eta, \mathbf{J}, M} > 0$  with the following properties: For any ideal  $\mathfrak{n} \in \mathcal{I}_{S \cup S(\mathfrak{q}), \eta}^-$  with  $N(\mathfrak{n}) > N_{\mathfrak{q}, S, l, \eta, \mathbf{J}, M}$  and  $\sum_{v \in S(\mathfrak{n})} \frac{\log q_v}{q_v} \leq M$ , there exists  $\pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n})$  such that*

- (i)  $\epsilon(1/2, \pi \otimes \eta) = -1$ ,
- (ii)  $L(1/2, \pi) \neq 0$  and  $L'(1/2, \pi \otimes \eta) \neq 0$ ,
- (iii)  $[\mathbb{Q}(\pi) : \mathbb{Q}] \geq C_{\mathfrak{q}} \sqrt{\log \log N(\mathfrak{n})}$ , and
- (iv)  $q_v^{\nu_v(\pi)/2} + q_v^{-\nu_v(\pi)/2} \in J_v$  for all  $v \in S$ .

We should note that the assumption (1.8) is a consequence of the Riemann hypothesis for the  $L$ -function  $L(s, \pi) L(s, \pi \otimes \eta)$ . Theorem 3 (Theorem 4) yields Hilbert cuspforms in arbitrarily large level with arbitrary large degree of the field of rationality, such that the central value of  $L$ -function and the central value (derivative) of its prescribed quadratic twist are nonzero simultaneously. Although we can expect a similar result for parallel weight 2 Hilbert cuspforms, our method does not work as it is for such low weight cases. In order to treat these interesting cases, the technique of Green's function as in [9] and [6] may be useful.

1.3. Let us describe a brief line of argument how we prove Theorems 1, 3 and 4, explaining the organization of this paper. In our previous work [7], we constructed the renormalized smoothed automorphic Green function  $\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha)$  as the value at  $\lambda = 0$  of an entire extension of some Poincaré series  $\hat{\Psi}_{\beta, \lambda}^l(\mathfrak{n}|\alpha)$  originally defined for  $\text{Re}(\lambda) > 1$ . Then we computed the period integral of  $\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha)$  along the diagonal split torus  $H$  adelicly in a very explicit form. In the present work, instead of the period integral, we introduce a certain integral transform  $\partial P_{\beta, \lambda}^\eta(\varphi)$  (see §2) for any continuous function  $\varphi$  on  $\text{PGL}(2, F) \backslash \text{PGL}(2, \mathbb{A})$  and a quadratic idele class character  $\eta$  of  $F^\times$ , depending on a complex parameter  $\lambda$  and a test function  $\beta$  for renormalization, whose constant term at  $\lambda = 0$  yields the derivative at  $s = 1/2$  of the period integral of  $\varphi |\det|_{\mathbb{A}}^{s-1/2}$  along  $H$ . The main step to have the second formula in Theorem 1 is to calculate  $\partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha))$  and its constant term at  $\lambda = 0$  in two different ways; the process is completely parallel to that in [7] for period integrals. In §2, after recalling the explicit formula of Hecke's zeta integrals for old forms ([5]) and calculating their derivatives, we prove a formula of  $\text{CT}_{\lambda=0} \partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha))$  written in terms of the spectral data of cuspidal representations in  $\Pi_{\text{cus}}^*(l, \mathfrak{n})$  (Proposition 8). In §3, closely following [7], we compute  $\partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha))$  according to the  $H(F) \times H(F)$ -double coset decomposition of  $\text{GL}(2, F)$ . By equating the

two expressions of  $\text{CT}_{\lambda=0} \partial P_{\beta,\lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha))$  obtained in §2 and §3, we get a kind of relative trace formula, which is stated in Theorem 17. The formula is not for our  $\text{ADL}^*(\mathbf{n})$  but for a similar average of  $L$ -values over all cuspidal representations  $\pi \in \Pi_{\text{cus}}(l, \mathbf{n})$ . We need to sieve out information on an average of only those  $\pi \in \Pi_{\text{cus}}(l, \mathbf{n})$  with exact conductor  $\mathbf{n}$ . For that purpose, we introduce a certain operation (see Definition 19), which we call the  $\mathcal{N}$ -transform, for any arithmetic function defined on a set of ideals. The first subsection of §4 is devoted to the study of the  $\mathcal{N}$ -transform. By applying the  $\mathcal{N}$ -transform of each term occurring in the formula (3.7), we deduce yet another formula (4.4), which relates the average  $\text{ADL}^*(\mathbf{n})$  to the sum of following terms: (i) the  $\mathcal{N}$ -transforms of  $\tilde{\mathbb{W}}_{\mathbf{u}}^\eta(l, \mathbf{n}|\alpha)$  and  $\mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha)$ , both of them occurring in the geometric side of (3.7), (ii) the  $L$ -value average  $\text{AL}^*(\mathbf{n})$  and (iii) the  $\mathcal{N}$ -transform of a certain term  $\text{AL}^{\partial w}(\mathbf{n})$  arising from the spectral side of (3.7). In §6, we analyze these terms separately and obtain an exact evaluation of the  $\mathcal{N}$ -transform of  $\tilde{\mathbb{W}}_{\mathbf{u}}^\eta(l, \mathbf{n}|\alpha)$  and estimations of the remaining terms, which lead us to the proof of the second formula of Theorem 1. In the proceeding §5, by applying the relative trace formula [7, Theorem 9.1] to the test function  $\alpha_a$ , we deduce the first asymptotic formula in Theorem 1, which is necessary to prove Theorem 3. In §7, we give the proof of Theorems 3 and 4. Actually, what we do there is to confirm that the argument of [4] for the classical modular forms still works with a minor modification in our setting. The analysis performed in §6 relies on explicit formulas of local integrals arising from  $\mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha)$  and  $\tilde{\mathbb{W}}_{\mathbf{u}}^\eta(l, \mathbf{n}|\alpha)$ ; the aim of §8 is to provide them. In the final section §9, we study a certain lattice sum to use it in the error term estimate in §5 and §8.

#### Notation :

- Given a condition  $P$ ,  $\delta(P)$  is 1 if  $P$  is true and 0 otherwise.
- $\mathbb{N}$  denotes the set of positive integers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .
- $F$  denotes a totally real number field.
  - $d_F = [F : \mathbb{Q}]$ .
  - $\mathfrak{o}$  : the integer ring of  $F$ .
  - $D_F$  : the absolute discriminant of  $F$ .
- $\Sigma_\infty$  (resp.  $\Sigma_{\text{fin}}$ ) : the set of infinite places (resp. finite places) of  $F$ . We set  $\Sigma_F = \Sigma_\infty \cup \Sigma_{\text{fin}}$ .
- $F_v$  : the completion of  $F$  at  $v \in \Sigma_F$ .
- $||_v$  : the normalized valuation of  $F_v$  for  $v \in \Sigma_F$ .
- For any  $v \in \Sigma_{\text{fin}}$ ,
  - $\mathfrak{o}_v$  : the integer ring of  $F_v$ .
  - $\varpi_v$  : a prime element of  $\mathfrak{o}_v$ .
  - $\mathfrak{p}_v$  : the prime ideal of  $\mathfrak{o}$  corresponding to  $v$ .
  - $q_v$  : the cardinality of the residue field  $\mathfrak{o}/\mathfrak{p}_v$  for  $v$ .
- $\mathbb{A}$  : the adele ring of  $F$ .
- For an ideal  $\mathfrak{m} \subset \mathfrak{o}$ ,
  - $S(\mathfrak{m}) = \{v \in \Sigma_{\text{fin}} \mid \text{ord}_v(\mathfrak{m}) > 0\}$ .
  - $S_k(\mathfrak{m}) = \{v \in S(\mathfrak{m}) \mid \text{ord}_v(\mathfrak{m}) = k\}$  for  $k \in \mathbb{N}$ .
  - $N(\mathfrak{m})$  : the absolute norm of  $\mathfrak{m}$ .
- $\eta = \prod_v \eta_v$  always denotes a quadratic idele class character of  $F^\times$ .

- $\mathfrak{f}_\eta = \prod_{v \in \Sigma_{\text{fin}}} \mathfrak{p}_v^{f(\eta_v)}$  : the conductor of  $\eta$ .
- $x_\eta^*$  : the finite idele such that  $x_{\eta,v}^* = \varpi_v^{-f(\eta_v)}$  for all  $v \in \Sigma_{\text{fin}}$ .
- $x_\eta$  : the adele such that  $x_{\eta,v} = 0$  for all  $v \in \Sigma_\infty$  and  $x_{\eta,\text{fin}} = x_\eta^*$ .
- $\epsilon(\eta)$  : the number of  $v \in \Sigma_\infty$  such that  $\eta_v(-1) = -1$ .
- $\tilde{\eta}$  : the character of the ideal group relatively prime to  $\mathfrak{f}_\eta$  defined by  $\tilde{\eta}(\mathfrak{p}_v) = \eta_v(\varpi_v)$ ,  $v \in \Sigma_{\text{fin}} - S(\mathfrak{f}_\eta)$ .
- $\mathfrak{n}$  always denotes an  $\mathfrak{o}$ -ideal prime to  $\mathfrak{f}_\eta$ .
- $H = \left\{ \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \mid t_1, t_2 \in \text{GL}(1) \right\}$ .
- $\mathbf{K}_{\text{fin}} = \prod_{v \in \Sigma_{\text{fin}}} \mathbf{K}_v$  with  $\mathbf{K}_v = \text{GL}(2, \mathfrak{o}_v)$  for  $v \in \Sigma_{\text{fin}}$ .
- $\mathbf{K}_0(\mathfrak{n}) = \prod_{v \in \Sigma_{\text{fin}}} \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$  with  $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{K}_v \mid c \in \mathfrak{n}\mathfrak{o}_v \right\}$  for  $v \in \Sigma_{\text{fin}}$ .
- $\pi$  : an irreducible cuspidal representation of  $\text{GL}(2, \mathbb{A})$  with trivial central character.
  - $\mathfrak{f}_\pi$  : the conductor of  $\pi$ .
  - $\{\pi_v\}_{v \in \Sigma_F}$  : a family of irreducible admissible representations of  $\text{GL}(2, F_v)$  such that  $\pi \cong \otimes_v \pi_v$ .
  - $S_\pi = \{v \in \Sigma_{\text{fin}} \mid \text{ord}_v(\mathfrak{f}_\pi) \geq 2\}$ .
  - For a place  $v \in \Sigma_{\text{fin}} - S(\mathfrak{f}_\pi)$ , the Satake parameter of  $\pi_v$  is denoted by  $A_v(\pi) = \text{diag}(q_v^{\nu_v(\pi)/2}, q_v^{-\nu_v(\pi)/2})$  with  $\nu_v(\pi) \in \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$ . We set  $\lambda_v(\pi) = \text{tr} A_v(\pi)$ .
  - $L^{S_\pi}(s, \pi; \text{Ad})$  : the adjoint square  $L$ -function of  $\pi$ , whose local factors above  $S_\pi$  are removed.
- $l = (l_v)_{v \in \Sigma_\infty}$  : an even weight, i.e., a system of positive even integers indexed by  $\Sigma_\infty$ . We set  $\underline{l} = \inf_{v \in \Sigma_\infty} l_v$ .
- $\Pi_{\text{cus}}(l, \mathfrak{n})$  : the set of all those  $\pi \cong \otimes_v \pi_v$  such that  $\pi_v$  is a discrete series representation of  $\text{PGL}(2, F_v)$  of weight  $l_v$  for any  $v \in \Sigma_\infty$  and  $\mathfrak{n} \subset \mathfrak{f}_\pi$ .
- $\Pi_{\text{cus}}^*(l, \mathfrak{n}) = \{\pi \in \Pi_{\text{cus}}(l, \mathfrak{n}) \mid \mathfrak{f}_\pi = \mathfrak{n}\}$ .
- $S$  : a finite subset of  $\Sigma_{\text{fin}} - S(\mathfrak{f}_\eta \mathfrak{n})$ .
  - $\mathfrak{X}_S$  : the complex manifold  $\prod_{v \in S} (\mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z})$ .
  - $\mathcal{A}_S = \otimes_{v \in S} \mathcal{A}_v$ , where for  $v \in \Sigma_{\text{fin}}$ ,  $\mathcal{A}_v$  denotes the space of holomorphic functions  $\alpha(s)$  in  $s \in \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$  such that  $\alpha(-s) = \alpha(s)$ .

## 2. SPECTRAL AVERAGE OF DERIVATIVES OF $L$ -SERIES : THE SPECTRAL SIDE

Let  $\mathcal{B}$  be the space of even entire functions  $\beta(z)$  on  $\mathbb{C}$  such that, for any finite interval  $I \subset \mathbb{R}$  and for any  $N > 0$ ,  $|\beta(\sigma + it)| \ll_{I,N} (1 + |t|)^{-N}$  for  $\sigma + it \in I + i\mathbb{R}$ . Given  $\beta \in \mathcal{B}$ ,  $t > 0$  and  $\lambda \in \mathbb{C}$ , we set

$$\beta_\lambda^{(1)}(t) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z + \lambda)^2} t^z dz,$$

where  $L_\sigma = \{z \in \mathbb{C} \mid \text{Re}(z) = \sigma\}$ . The defining integral is independent of the choice of  $\sigma > -\text{Re}(\lambda)$ . By the residue theorem,

$$(2.1) \quad \text{CT}_{\lambda=0} \{\beta_\lambda^{(1)}(t) - \beta_\lambda^{(1)}(t^{-1})\} = \beta(0) \log t.$$

In the same way as [9, Lemma 7.1], we have the estimate

$$(2.2) \quad |\beta_\lambda^{(1)}(t)| \ll_\sigma \inf\{t^\sigma, t^{-\text{Re}(\lambda)}\} \log t, \quad t > 0, \sigma > -\text{Re}(\lambda).$$

**Definition 5.** For a cusp form  $\varphi$  on  $\mathrm{PGL}(2, \mathbb{A})$ , set

$$\partial P_{\beta, \lambda}^{\eta}(\varphi) = \int_{F^{\times} \backslash \mathbb{A}^{\times}} \varphi \left( \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_{\eta} \\ 0 & 1 \end{bmatrix} \right) \eta(tx_{\eta}^*) \{ \beta_{\lambda}^{(1)}(|t|_{\mathbb{A}}) - \beta_{\lambda}^{(1)}(|t|_{\mathbb{A}}^{-1}) \} d^{\times} t, \quad \mathrm{Re}(\lambda) \gg 0.$$

By (2.2), the integral  $\partial P_{\beta, \lambda}^{\eta}(\varphi)$  is absolutely convergent for  $\lambda \in \mathbb{C}$  and the function  $\lambda \mapsto \partial P_{\beta, \lambda}^{\eta}(\varphi)$  is entire on  $\mathbb{C}$ . Therefore, (2.1) gives us the formula

$$\mathrm{CT}_{\lambda=0} \partial P_{\beta, \lambda}^{\eta}(\varphi) = \int_{F^{\times} \backslash \mathbb{A}^{\times}} \varphi \left( \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_{\eta} \\ 0 & 1 \end{bmatrix} \right) \eta(tx_{\eta}^*) \log |t|_{\mathbb{A}} d^{\times} t \beta(0) = \frac{d}{ds} Z^*(s, \eta, \varphi) \Big|_{s=1/2} \beta(0).$$

Here  $Z^*(s, \eta, \varphi)$  is the modified global zeta integral considered in [9, 2.6.2], [5, §4], [6, §2.1] and [7, §6.3].

2.1. For  $j \in \mathbb{N}_0$ , a place  $v \in \Sigma_{\mathrm{fin}}$ , an irreducible admissible representation  $\pi_v$  of  $\mathrm{PGL}(2, F_v)$  and for a character  $\eta_v$  of  $F_v^{\times}$  such that  $\eta_v^2 = 1$ , we define a polynomial of  $X$  by setting  $Q_{j,v}^{\pi_v}(\eta_v, X) =$

$$(2.3) \quad \begin{cases} 1, & (j = 0), \\ \eta_v(\varpi_v)X - Q(\pi_v), & (c(\pi_v) = 0, j = 1), \\ \eta_v(\varpi_v)^{j-1} X^{j-1} (\eta_v(\varpi_v)X - q_v^{-1} \chi_v(\varpi_v)^{-1}), & (c(\pi_v) = 1, j \geq 1), \\ q_v^{-1} \eta_v(\varpi_v)^{j-2} X^{j-2} (a_v q_v^{1/2} \eta_v(\varpi_v)X - 1)(a_v^{-1} q_v^{1/2} \eta_v(\varpi_v)X - 1), & (c(\pi_v) = 0, j \geq 2), \\ \eta_v(\varpi_v)^j X^j, & (c(\pi_v) \geq 2, j \geq 1), \end{cases}$$

(cf. [5, Corollary 19]), where

$$Q(\pi_v) = (a_v + a_v^{-1}) / (q_v^{1/2} + q_v^{-1/2}) \quad \text{with } a_v^{\pm} \text{ the Satake parameter of } \pi_v \text{ if } c(\pi_v) = 0,$$

and  $\chi_v$  is the unramified character of  $F_v^{\times}$  such that  $\pi_v \cong \sigma(\chi_v | \cdot|_v^{1/2}, \chi_v | \cdot|_v^{-1/2})$  if  $c(\pi_v) = 1$ . For  $\pi \in \Pi_{\mathrm{cus}}(l, \mathbf{n})$ , we set

$$Q_{\pi, \eta, \rho}(s) = \prod_{v \in S(\mathfrak{nf}_{\pi}^{-1})} Q_{\rho(v), v}^{\pi_v}(\eta_v, q_v^{1/2-s}), \quad \rho \in \Lambda_{\pi}(\mathbf{n}),$$

where  $\Lambda_{\pi}(\mathbf{n})$  denotes the set  $\{\rho : \Sigma_{\mathrm{fin}} \rightarrow \mathbb{N}_0 \mid 0 \leq \rho(v) \leq \mathrm{ord}_v(\mathfrak{nf}_{\pi}^{-1}) \ (\forall v \in \Sigma_{\mathrm{fin}})\}$ . We recall here an explicit formula of the modified zeta integral  $Z^*(s, \eta, \varphi_{\pi, \rho})$  for the basis  $\{\varphi_{\pi, \rho}\}$  of  $V_{\pi}[\tau_l]^{\mathbf{K}_0(\mathbf{n})}$  ([5, Proposition 20] and [7, Proposition 6.1]):

$$(2.4) \quad Z^*(s, \eta, \varphi_{\pi, \rho}) = D_F^{s-1/2} (-1)^{\epsilon(\eta)} \mathcal{G}(\eta) Q_{\pi, \eta, \rho}(s) L(s, \pi \otimes \eta)$$

for any  $\pi \in \Pi_{\mathrm{cus}}(l, \mathbf{n})$  and  $\rho \in \Lambda_{\pi}(\mathbf{n})$ .

2.2. Let  $\pi \in \Pi_{\mathrm{cus}}(l, \mathbf{n})$  and  $\rho \in \Lambda_{\pi}(\mathbf{n})$ . For a complex parameter  $z$ , we set

$$(2.5) \quad w_{\mathbf{n}}^{\eta}(\pi; z) = \sum_{\rho \in \Lambda_{\pi}(\mathbf{n})} \prod_{v \in S(\mathfrak{nf}_{\pi}^{-1})} \overline{Q_{\rho(v), v}^{\pi_v}(\mathbf{1}, \mathbf{1})} Q_{\rho(v), v}^{\pi_v}(\eta_v, q_v^{1/2-z}) / \tau_{\pi_v}(\rho(v), \rho(v))$$

$$(2.6) \quad = \prod_{v \in S(\mathfrak{nf}_{\pi}^{-1})} r^{(z)}(\pi_v, \eta_v)$$



with

$$r^{(z)}(\pi_v, \eta_v) = \sum_{j=0}^{\text{ord}_v(\mathfrak{nf}_\pi^{-1})} \overline{Q_{j,v}^{\pi_v}(\mathbf{1}, 1)} Q_{j,v}^{\pi_v}(\eta_v, q_v^{1/2-z}) / \tau_{\pi_v}(j, j)$$

Here  $Q_{j,v}^{\pi_v}(\eta_v, X)$  is the polynomial defined by (2.3), and  $\tau_{\pi_v}(j, j)$  is given by [5, Corollary 12, Corollary 16 and Lemma 3] as

$$(2.7) \quad \tau_{\pi_v}(j, j) = \begin{cases} 1, & (j = 0 \text{ or } c(\pi_v) \geq 2), \\ 1 - Q(\pi_v)^2, & (c(\pi_v) = 0, j = 1), \\ 1 - q_v^{-2}, & (c(\pi_v) = 1, j \geq 1), \\ (1 - Q(\pi_v)^2)(1 - q_v^{-2}), & (c(\pi_v) = 0, j \geq 2). \end{cases}$$

Here is the explicit determination of  $r^{(z)}(\pi_v, \eta_v)$ .

**Lemma 6.** *Let  $v \in S(\mathfrak{nf}_\pi^{-1})$  and set  $k_v = \text{ord}_v(\mathfrak{nf}_\pi^{-1})$ ,  $X = q_v^{1/2-z}$ . Suppose  $\eta_v(\varpi_v) = -1$ . Then we have*

$$r^{(z)}(\pi_v, \eta_v) = \begin{cases} \frac{1-X}{1+Q(\pi_v)} + \frac{(1+a_v q_v^{1/2} X)(1+a_v^{-1} q_v^{1/2} X)}{(q_v-1)(1+Q(\pi_v))} \frac{1-(-X)^{k_v-1}}{1+X}, & (c(\pi_v) = 0), \\ 1 - \frac{X+q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)} \frac{1-(-1)^{k_v} X^{k_v}}{1+X}, & (c(\pi_v) = 1), \\ \frac{1+(-1)^{k_v} X^{k_v+1}}{1+X}, & (c(\pi_v) \geq 2). \end{cases}$$

Suppose  $\eta_v(\varpi_v) = 1$ . Then we have

$$r^{(z)}(\pi_v, \eta_v) = \begin{cases} \frac{1+X}{1+Q(\pi_v)} + \frac{(1-a_v q_v^{1/2} X)(1-a_v^{-1} q_v^{1/2} X)}{(q_v-1)(1+Q(\pi_v))} \sum_{j=2}^{k_v} X^{j-2}, & (c(\pi_v) = 0), \\ 1 + \frac{X-q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)} (\sum_{j=1}^{k_v} X^j), & (c(\pi_v) = 1), \\ \sum_{j=0}^{k_v} X^j, & (c(\pi_v) \geq 2). \end{cases}$$

*Proof.* From (2.3) and (2.7), we obtain the result by a direct computation.  $\square$

We abbreviate  $r^{(1/2)}(\pi_v, \eta_v)$  to  $r(\pi_v, \eta_v)$ . Define

$$w_\pi^\eta(\pi) = w_\pi^\eta(\pi; 1/2), \quad \partial w_\pi^\eta(\pi) = \left( \frac{d}{dz} \right)_{z=1/2} w_\pi^\eta(\pi; z).$$

Note that the first quantity  $w_\pi^\eta(\pi)$  is the same one as in [6, Lemma 12] and [7, Lemma 6.2]. From Lemma 6, the second quantity  $\partial w_\pi^\eta(\pi)$  is also evaluated explicitly.

**Corollary 7.** *Set  $\partial r(\pi_v, \eta_v) = \frac{-1}{\log q_v} \left( \frac{d}{dz} \right)_{z=1/2} r^{(z)}(\pi_v, \eta_v)$ . When  $\eta_v(\varpi_v) = -1$ ,*

$$\partial r(\pi_v, \eta_v) = \begin{cases} \frac{-1}{1+Q(\pi_v)} + \frac{1+(-1)^{k_v}}{2} \frac{2q_v+(q_v+1)Q(\pi_v)}{(q_v-1)(1+Q(\pi_v))} + \frac{(-1)^{k_v}(2k_v-3)-1}{4} \frac{q_v+1}{q_v-1}, & (c(\pi_v) = 0), \\ -\frac{1-(-1)^{k_v}}{2} \frac{1}{1+q_v^{-1}\chi_v(\varpi_v)} + \frac{1+(-1)^{k_v}(2k_v-1)}{4}, & (c(\pi_v) = 1), \\ \frac{(-1)^{k_v}(2k_v+1)-1}{4}, & (c(\pi_v) \geq 2). \end{cases}$$



When  $\eta_v(\varpi_v) = 1$ ,

$$\partial r(\pi_v, \eta_v) = \begin{cases} \frac{1}{1+Q(\pi_v)} + (k_v - 1) \frac{2q_v - (q_v+1)Q(\pi_v)}{(q_v-1)(1+Q(\pi_v))} + \frac{(k_v-2)(k_v-1)}{2} \frac{(q_v+1)(1-Q(\pi_v))}{(q_v-1)(1+Q(\pi_v))}, & (c(\pi_v) = 0), \\ \frac{k_v}{\frac{1+q_v^{-1}\chi_v(\varpi_v)}{2}} + \frac{1-q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)} \frac{k_v(k_v+1)}{2}, & (c(\pi_v) = 1), \\ \frac{k_v(k_v+1)}{2}, & (c(\pi_v) \geq 2). \end{cases}$$

2.3. Depending on a function  $\alpha \in \mathcal{A}_S$ , we have constructed a cusp form denoted by  $\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha)$  in [7, 6.5.3]. Recall that it has the expression

(2.8)

$$\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha; g) = \frac{(-1)^{\#S} \{ \prod_{v \in \Sigma_\infty} 2^{l_v-1} \} C_l(0) D_F^{-1/2}}{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathbf{n})]} \sum_{\pi \in \Pi_{\text{cus}}(l, \mathbf{n})} \sum_{\rho \in \Lambda_\pi(\mathbf{n})} \alpha(\nu_S(\pi)) \frac{\overline{Z^*(1/2, \mathbf{1}, \varphi_{\pi, \rho})}}{\|\varphi_{\pi, \rho}\|^2} \varphi_{\pi, \rho}(g).$$

**Proposition 8.** *We have*

$$\begin{aligned} \text{CT}_{\lambda=0} \partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha)) &= (-1)^{\#S} \left\{ \prod_{v \in \Sigma_\infty} 2^{l_v-1} \right\} C_l(0) D_F^{-1} [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathbf{n})]^{-1} (-1)^{\epsilon(\eta)} \mathcal{G}(\eta) \\ &\quad \times \left[ \sum_{\pi \in \Pi_{\text{cus}}(l, \mathbf{n})} (\log D_F) w_{\mathbf{n}}^\eta(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\|\varphi_\pi^{\text{new}}\|^2} \alpha(\nu_S(\pi)) \right. \\ &\quad + \sum_{\pi \in \Pi_{\text{cus}}(l, \mathbf{n})} \partial w_{\mathbf{n}}^\eta(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\|\varphi_\pi^{\text{new}}\|^2} \alpha(\nu_S(\pi)) \\ &\quad \left. + \sum_{\pi \in \Pi_{\text{cus}}(l, \mathbf{n})} w_{\mathbf{n}}^\eta(\pi) \frac{L(1/2, \pi) L'(1/2, \pi \otimes \eta)}{\|\varphi_\pi^{\text{new}}\|^2} \alpha(\nu_S(\pi)) \right] \beta(0). \end{aligned}$$

*Proof.* Since the spectral expansion (2.8) is a finite sum, we have

$$\begin{aligned} \text{CT}_{\lambda=0} \partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha)) &= (-1)^{\#S} \prod_{v \in \Sigma_\infty} 2^{l_v-1} C_l(0) D_F^{-1/2} [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathbf{n})]^{-1} \\ &\quad \times \sum_{\pi \in \Pi_{\text{cus}}(l, \mathbf{n})} \sum_{\rho \in \Lambda_\pi(\mathbf{n})} \alpha(\nu_S(\pi)) \frac{\overline{Z^*(1/2, \mathbf{1}, \varphi_{\pi, \rho})}}{\|\varphi_{\pi, \rho}\|^2} \frac{d}{ds} Z^*(s, \eta, \varphi_{\pi, \rho}) \Big|_{s=1/2} \beta(0). \end{aligned}$$

By virtue of [7, Proposition 6.1] and (2.4), we have

$$\begin{aligned} &\sum_{\rho \in \Lambda_\pi(\mathbf{n})} \frac{\overline{Z^*(1/2, \mathbf{1}, \varphi_{\pi, \rho})}}{\|\varphi_{\pi, \rho}\|^2} \frac{d}{ds} Z^*(s, \eta, \varphi_{\pi, \rho}) \Big|_{s=1/2} \\ &= \sum_{\rho \in \Lambda_\pi(\mathbf{n})} \frac{1}{\|\varphi_{\pi, \rho}\|^2} D_F^{-1/2} Q_{\pi, \mathbf{1}, \rho}(1/2) L(1/2, \pi) (\log D_F) \mathcal{G}(\eta) Q_{\pi, \eta, \rho}(1/2) L(1/2, \pi \otimes \eta) \\ &\quad + \sum_{\rho \in \Lambda_\pi(\mathbf{n})} \frac{1}{\|\varphi_{\pi, \rho}\|^2} D_F^{-1/2} Q_{\pi, \mathbf{1}, \rho}(1/2) L(1/2, \pi) \mathcal{G}(\eta) (Q_{\pi, \eta, \rho})'(1/2) L(1/2, \pi \otimes \eta) \\ &\quad + \sum_{\rho \in \Lambda_\pi(\mathbf{n})} \frac{1}{\|\varphi_{\pi, \rho}\|^2} D_F^{-1/2} Q_{\pi, \mathbf{1}, \rho}(1/2) L(1/2, \pi) \mathcal{G}(\eta) Q_{\pi, \eta, \rho}(1/2) L'(1/2, \pi \otimes \eta) \\ &= (\log D_F) D_F^{-1/2} \mathcal{G}(\eta) w_{\mathbf{n}}^\eta(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\|\varphi_\pi^{\text{new}}\|^2} \end{aligned}$$

$$\begin{aligned}
& + \left\{ \sum_{\rho \in \Lambda_\pi(\mathfrak{n})} \left( \prod_{v \in S(\mathfrak{nf}_\pi^{-1})} \frac{Q_{\rho(v),v}^{\pi_v}(\mathbf{1}_v, 1)}{\tau_{\pi_v}(\rho(v), \rho(v))} \right) (Q_{\pi, \eta, \rho})'(1/2) \right\} D_F^{-1/2} \mathcal{G}(\eta) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\|\varphi_\pi^{\text{new}}\|^2} \\
& + D_F^{-1/2} \mathcal{G}(\eta) w_\pi^\eta(\pi) \frac{L(1/2, \pi) L'(1/2, \pi \otimes \eta)}{\|\varphi_\pi^{\text{new}}\|^2}.
\end{aligned}$$

By the first expression (2.5) of  $w_\pi^\eta(\pi, z)$ , we have

$$\partial w_\pi^\eta(\pi) = \sum_{\rho \in \Lambda_\pi(\mathfrak{n})} \left\{ \prod_{v \in S(\mathfrak{nf}_\pi^{-1})} \frac{\overline{Q_{\rho(v),v}^{\pi_v}(\mathbf{1}_v, 1)}}{\tau_{\pi_v}(\rho(v), \rho(v))} \right\} (Q_{\pi, \eta, \rho})'(1/2).$$

Thus we are done.  $\square$

### 3. SPECTRAL AVERAGE OF DERIVATIVES OF $L$ -SERIES: THE GEOMETRIC SIDE

Recall that the function  $\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha)$  has another expansion coming from double cosets  $H(F) \backslash \text{GL}(2, F) / H(F)$  ([7, §7]):

(3.1)

$$\hat{\Psi}_{\text{reg}}^l(\mathfrak{n}|\alpha; \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_\eta \\ 0 & 1 \end{bmatrix}) = (1 + i^l \delta(\mathfrak{n} = \mathfrak{o})) J_{\text{id}}(\alpha; t) + J_{\text{u}}(\alpha; t) + J_{\bar{\text{u}}}(\alpha; t) + J_{\text{hyp}}(\alpha; t), \quad t \in \mathbb{A}^\times,$$

where the terms in the right-hand side are defined in [7, Lemma 7.1, Lemma 7.2 and Lemma 7.11]. For  $\mathfrak{h} \in \{\text{id}, \text{u}, \bar{\text{u}}, \text{hyp}\}$ , we consider the “orbital integrals”

$$\mathbb{W}_{\mathfrak{h}}^\eta(\beta, \lambda; \alpha) = \int_{F^\times \backslash \mathbb{A}^\times} J_{\mathfrak{h}}(\alpha; t) \{ \beta_\lambda^{(1)}(|t|_{\mathbb{A}}) - \beta_\lambda^{(1)}(|t|_{\mathbb{A}}^{-1}) \} \eta(tx_\eta^*) d^\times t$$

for  $\alpha \in \mathcal{A}$ ,  $\beta \in \mathcal{B}$  and  $\lambda \in \mathbb{C}$  such that  $\text{Re}(\lambda) > 1$ . We shall show that these integrals converge absolutely individually when  $\text{Re}(\lambda) > 1$  and admit an analytic continuation in a neighborhood of  $\lambda = 0$ .

**Lemma 9.** *Let  $\lambda$  and  $w$  be complex numbers such that  $\text{Re}(w) < \text{Re}(\lambda)$ . Let  $\xi$  be an idele class character of  $F^\times$ . Then, we have*

$$\int_{F^\times \backslash \mathbb{A}^\times} \beta_\lambda^{(1)}(|t|_{\mathbb{A}}) \xi(t) |t|_{\mathbb{A}}^w d^\times t = \delta_{\xi, 1} \text{vol}(F^\times \backslash \mathbb{A}^1) \frac{\beta(-w)}{(\lambda - w)^2}.$$

*Proof.* The proof is given in the same way as [9, Lemma 7.6].  $\square$

**Lemma 10.** *For  $\text{Re}(\lambda) > 0$ , the integral  $\mathbb{W}_{\text{id}}^\eta(\beta, \lambda; \alpha)$  converges absolutely and  $\mathbb{W}_{\text{id}}^\eta(\beta, \lambda; \alpha) = 0$ .*

*Proof.* This follows immediately from Lemma 9, since  $J_{\text{id}}(\alpha; t)$  is independent of the variable  $t$  ([7, Lemma 7.1]).  $\square$

Assume that  $q(\text{Re}(\mathfrak{s})) > \text{Re}(\lambda) > \sigma > 1$ . Set

$$\begin{aligned}
V_{0, \eta}^\pm(\lambda; \mathfrak{s}) &= \frac{1}{2\pi i} \int_{L_{\mp\sigma}} \frac{\beta(z)}{(z + \lambda)^2} \int_{\mathbb{A}^\times} \Psi_l^{(0)}(\mathfrak{n}|\mathfrak{s}; \begin{bmatrix} 1 & t^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_\eta \\ 0 & 1 \end{bmatrix}) \eta(tx_\eta^*) |t|_{\mathbb{A}}^{\pm z} d^\times t dz, \\
V_{1, \eta}^\pm(\lambda; \mathfrak{s}) &= \frac{1}{2\pi i} \int_{L_{\mp\sigma}} \frac{\beta(z)}{(z + \lambda)^2} \int_{\mathbb{A}^\times} \Psi_l^{(0)}(\mathfrak{n}|\mathfrak{s}; \begin{bmatrix} 1 & 0 \\ t^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x_\eta & 1 \end{bmatrix} w_0) \eta(tx_\eta^*) |t|_{\mathbb{A}}^{\pm z} d^\times t dz
\end{aligned}$$

and

$$\Upsilon_S^\eta(z; \mathbf{s}) = \prod_{v \in S} (1 - \eta_v(\varpi_v) q_v^{-(z+(s_v+1)/2)})^{-1} (1 - q_v^{(s_v+1)/2})^{-1},$$

$$\Upsilon_{S,l}^\eta(z; \mathbf{s}) = D_F^{-1/2} \{ \#(\mathfrak{o}/\mathfrak{f}_\eta)^\times \}^{-1} \left\{ \prod_{v \in \Sigma_\infty} \frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z+\epsilon_v)\Gamma(l_v/2)} i^{\epsilon_v} \cos\left(\frac{\pi}{2}(-z+\epsilon_v)\right) \right\} \Upsilon_S^\eta(z; \mathbf{s}).$$

**Lemma 11.** *The double integrals  $V_{j,\eta}^\pm(\lambda; \mathbf{s})$  converge absolutely and*

$$V_{0,\eta}^\pm(\lambda; \mathbf{s}) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} N(\mathfrak{f}_\eta)^{\mp z} L(\mp z, \eta) (-1)^{\epsilon(\eta)} \Upsilon_{S,l}^\eta(\pm z; \mathbf{s}) dz$$

and

$$V_{1,\eta}^\pm(\lambda; \mathbf{s}) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} N(\mathfrak{f}_\eta)^{\mp z} N(\mathbf{n})^{\pm z} \tilde{\eta}(\mathbf{n}) \delta(\mathbf{n} = \mathfrak{o}) L(\mp z, \eta) i^{\tilde{l}} \Upsilon_{S,l}^\eta(\pm z; \mathbf{s}) dz.$$

*Proof.* As in [7, Lemma 8.2], we exchange the order of integrals and compute the  $t$ -integrals first. Since  $\eta \neq 1$ , the integrands in the remaining contour integrals in  $z$  are holomorphic on  $|\operatorname{Re}(z)| < \sigma$ ; thus we can shift the contour  $L_{-\sigma}$  to  $L_\sigma$  for  $V_{0,\eta}^+$  and  $V_{1,\eta}^+$ .  $\square$

**Lemma 12.** *The integral  $\mathbb{W}_u^\eta(\beta, \lambda; \alpha)$  has an analytic continuation to  $\mathbb{C}$  as a function in  $\lambda$ . The constant term of  $\mathbb{W}_u^\eta(\beta, \lambda; \alpha)$  at  $\lambda = 0$  equals  $\mathbb{W}_u^\eta(l, \mathbf{n}|\alpha)\beta(0)$  with*

$$\mathbb{W}_u^\eta(l, \mathbf{n}|\alpha) = (-1)^{\epsilon(\eta)} \mathcal{G}(\eta) D_F^{1/2} (1 + (-1)^{\epsilon(\eta)} \tilde{\eta}(\mathbf{n}) i^{\tilde{l}} \delta(\mathbf{n} = \mathfrak{o})) \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \mathfrak{W}_S^\eta(l, \mathbf{n}|\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s}),$$

where  $\Upsilon_S^\eta(\mathbf{s}) = \Upsilon_S^\eta(0; \mathbf{s})$  and

$$\begin{aligned} \mathfrak{W}_S^\eta(\mathbf{s}) = & \pi^{\epsilon(\eta)} \Upsilon_S^\eta(\mathbf{s}) L(1, \eta) \left\{ \log D_F + \frac{L'(1, \eta)}{L(1, \eta)} \right. \\ & \left. + \sum_{v \in \Sigma_\infty} \left( \sum_{k=1}^{l_v/2-1} \frac{1}{k} - \frac{1}{2} \log \pi - \frac{1}{2} C_{\text{Euler}} - \delta_{\epsilon_v, 1} \log 2 \right) + \sum_{v \in S} \frac{\log q_v}{1 - \eta_v(\varpi_v) q_v^{(s_v+1)/2}} \right\}. \end{aligned}$$

*Proof.* From [7, Lemma 7.2], we have the expression

$$\mathbb{W}_u^\eta(\beta, \lambda; \alpha) = \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \{V_{0,\eta}^+(\lambda; \mathbf{s}) - V_{0,\eta}^-(\lambda; \mathbf{s}) + V_{1,\eta}^+(\lambda; \mathbf{s}) - V_{1,\eta}^-(\lambda; \mathbf{s})\} \alpha(\mathbf{s}) d\mu_S(\mathbf{s}).$$

By Lemma 11, the right-hand side becomes

$$\begin{aligned} & ((-1)^{\epsilon(\eta)} + i^{\tilde{l}} \delta(\mathbf{n} = \mathfrak{o})) \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} \{N(\mathfrak{f}_\eta)^{-z} L(-z, \eta) \Upsilon_{S,l}^\eta(z; \mathbf{s}) \\ & - N(\mathfrak{f}_\eta)^z L(z, \eta) \Upsilon_{S,l}^\eta(-z; \mathbf{s})\} dz \alpha(\mathbf{s}) d\mu_S(\mathbf{s}). \end{aligned}$$

which is holomorphic on  $\text{Re}(\lambda) > -\sigma$ . Since  $\sigma > 1$  is arbitrary, this gives an analytic continuation of  $\mathbb{W}_u^\eta(\beta, \lambda; \alpha)$  to  $\mathbb{C}$  and yields the equality

$$\begin{aligned} & \text{CT}_{\lambda=0} \mathbb{W}_u^\eta(\beta, \lambda; \alpha) \\ &= \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \left( \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{z^2} \{f_u(z) - f_u(-z)\} dz \right) \alpha(\mathbf{s}) d\mu_S(\mathbf{s}) \\ &= ((-1)^{\epsilon(\eta)} + i^{\tilde{l}} \delta(\mathbf{n} = \mathbf{o})) \text{Res}_{z=0} \left( \frac{\beta(z)}{z^2} f_u(z) \right) \\ &= ((-1)^{\epsilon(\eta)} + i^{\tilde{l}} \delta(\mathbf{n} = \mathbf{o})) (\text{CT}_{z=0} \frac{f_u(z)}{z} \beta(0) + \frac{1}{2} \text{Res}_{z=0} f_u(z) \beta''(0)), \end{aligned}$$

where  $f_u(z) = N(\mathbf{f}_\eta)^{-z} L(-z, \eta) \Upsilon_{S,l}^\eta(z; \mathbf{s})$ . Since  $\eta$  is nontrivial, by the functional equation

$$L(s, \eta) = i^{\epsilon(\eta)} D_F^{1-s} N(\mathbf{f}_\eta)^{-s} \#((\mathbf{o}/\mathbf{f}_\eta)^\times) \mathcal{G}(\eta) L(1-s, \eta),$$

$f_u(z)$  is holomorphic at  $z = 0$ . Thus,

$$\begin{aligned} & \text{CT}_{z=0} \frac{f_u(z)}{z} = \lim_{z \rightarrow 0} \frac{f_u(z) - f_u(0)}{z} \\ &= -(\log N(\mathbf{f}_\eta)) L(0, \eta) \Upsilon_{S,l}^\eta(0; \mathbf{s}) - L'(0, \eta) \Upsilon_{S,l}^\eta(0; \mathbf{s}) + L(0, \eta) (\Upsilon_{S,l}^\eta)'(0; \mathbf{s}) \\ &= i^{\epsilon(\eta)} \mathcal{G}(\eta) D_F^{1/2} \tilde{\Upsilon}_{S,l}^\eta(0; \mathbf{s}) \{-L(1, \eta) \log N(\mathbf{f}_\eta) + L(1, \eta) \log(D_F N(\mathbf{f}_\eta)) + L'(1, \eta) + L(1, \eta) \frac{d}{dz} \log \tilde{\Upsilon}_{S,l}(z; \mathbf{s})|_{z=0}\} \\ &= \mathcal{G}(\eta) D_F^{1/2} \pi^{\epsilon(\eta)} \Upsilon_S^\eta(\mathbf{s}) \{L(1, \eta) \log D_F + L'(1, \eta) + L(1, \eta) \frac{d}{dz} \log \tilde{\Upsilon}_{S,l}(z; \mathbf{s})|_{z=0}\}, \end{aligned}$$

where  $\tilde{\Upsilon}_{S,l}^\eta(z; \mathbf{s}) = D_F^{1/2} \#((\mathbf{o}/\mathbf{f}_\eta)^\times) \Upsilon_{S,l}^\eta(z; \mathbf{s})$ . Furthermore,

$$\begin{aligned} \frac{d}{dz} \log \tilde{\Upsilon}_{S,l}(z; \mathbf{s})|_{z=0} &= \sum_{v \in \Sigma_\infty} \left( \psi(l_v/2) - \frac{1}{2} \log \pi + \frac{1}{2} \psi \left( \frac{-z + \epsilon_v}{2} \right) - \psi(-z) + \frac{\pi}{2} \tan \frac{\pi}{2} (-z + \epsilon_v) \right) \Big|_{z=0} \\ &\quad + \sum_{v \in S} \frac{\log q_v}{1 - \eta_v(\varpi_v) q_v^{(s_v+1)/2}}. \end{aligned}$$

Here, by  $\psi(1) = -C_{\text{Euler}}$ ,  $\psi(1/2) = -C_{\text{Euler}} - 2 \log 2$  and  $\frac{d}{dt} (t \cot t)|_{t=0} = 0$ , we have

$$\frac{1}{2} \psi \left( \frac{-z + \epsilon_v}{2} \right) - \psi(-z) + \frac{\pi}{2} \tan \frac{\pi}{2} (-z + \epsilon_v) \Big|_{z=0} = \begin{cases} \frac{1}{2} C_{\text{Euler}} & (\epsilon_v = 0), \\ \frac{1}{2} \psi \left( \frac{1}{2} \right) - \psi(1) = \frac{1}{2} C_{\text{Euler}} - \log 2 & (\epsilon_v = 1). \end{cases}$$

□

Assume that  $q(\text{Re}(\mathbf{s})) > \text{Re}(\lambda) > \sigma > 1$ . Set

$$\begin{aligned} \tilde{V}_{1,\eta}^\pm(\lambda; \mathbf{s}) &= \frac{1}{2\pi i} \int_{L_{\pm\sigma}} \frac{\beta(z)}{(z + \lambda)^2} \int_{\mathbb{A}^\times} \Psi_l^{(0)}(\mathbf{n}|\mathbf{s}; \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 & x_\eta \\ 0 & 1 \end{bmatrix}) \eta(tx_\eta^*) |t|_{\mathbb{A}}^{\pm z} d^\times t dz, \\ \tilde{V}_{0,\eta}^\pm(\lambda; \mathbf{s}) &= \frac{1}{2\pi i} \int_{L_{\pm\sigma}} \frac{\beta(z)}{(z + \lambda)^2} \int_{\mathbb{A}^\times} \Psi_l^{(0)}(\mathbf{n}|\mathbf{s}; \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x_\eta & 1 \end{bmatrix} w_0) \eta(tx_\eta^*) |t|_{\mathbb{A}}^{\pm z} d^\times t dz. \end{aligned}$$

In the same way as Lemma 11, we obtain

**Lemma 13.** *The double integrals  $\tilde{V}_{j,\eta}^\pm(\lambda; \mathbf{s})$  converge absolutely and*

$$\begin{aligned}\tilde{V}_{1,\eta}^\pm(\lambda; \mathbf{s}) &= \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} N(\mathbf{f}_\eta)^{\mp z} N(\mathbf{n})^{\mp z} \tilde{\eta}(\mathbf{n}) L(\pm z, \eta) \Upsilon_{S,l}^\eta(\mp z; \mathbf{s}) dz, \\ \tilde{V}_{0,\eta}^\pm(\lambda; \mathbf{s}) &= \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} N(\mathbf{f}_\eta)^{\mp z} \delta(\mathbf{n} = \mathbf{o}) L(\pm z, \eta) (-1)^{\epsilon(\eta)} i^{\tilde{l}} \Upsilon_{S,l}^\eta(\mp z; \mathbf{s}) dz.\end{aligned}$$

**Lemma 14.** *The integral  $\mathbb{W}_{\bar{\mathbf{u}}}^\eta(\beta, \lambda; \alpha)$  converges absolutely on  $\operatorname{Re}(\lambda) > 1$  and has an analytic continuation to  $\mathbb{C}$  as a function in  $\lambda$ . The constant term of  $\mathbb{W}_{\bar{\mathbf{u}}}^\eta(\beta, \lambda; \alpha)$  at  $\lambda = 0$  equals  $\mathbb{W}_{\bar{\mathbf{u}}}^\eta(l, \mathbf{n}|\alpha)\beta(0)$  with*

$$\mathbb{W}_{\bar{\mathbf{u}}}^\eta(l, \mathbf{n}|\alpha) = (-1)^{\epsilon(\eta)} \mathcal{G}(\eta) D_F^{1/2} ((-1)^{\epsilon(\eta)} \tilde{\eta}(\mathbf{n}) + i^{\tilde{l}} \delta(\mathbf{n} = \mathbf{o})) \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \mathfrak{W}_{S,\bar{\mathbf{u}}}^\eta(l, \mathbf{n}|\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s}),$$

where

$$\mathfrak{W}_{S,\bar{\mathbf{u}}}^\eta(l, \mathbf{n}|\mathbf{s}) = -\pi^{\epsilon(\eta)} \Upsilon_S^\eta(\mathbf{s}) L(1, \eta) \log(N(\mathbf{n})N(\mathbf{f}_\eta)^2) - \mathfrak{W}_{S,\mathbf{u}}^\eta(l, \mathbf{n}|\mathbf{s}).$$

*Proof.* From [7, Lemma 7.2], we have the expression

$$\mathbb{W}_{\bar{\mathbf{u}}}^\eta(\beta, \lambda; \alpha) = \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \{ \tilde{V}_{0,\eta}^+(\lambda; \mathbf{s}) - \tilde{V}_{0,\eta}^-(\lambda; \mathbf{s}) + \tilde{V}_{1,\eta}^+(\lambda; \mathbf{s}) - \tilde{V}_{1,\eta}^-(\lambda; \mathbf{s}) \} \alpha(\mathbf{s}) d\mu_S(\mathbf{s}).$$

By Lemma 13, the right-hand side becomes

$$\begin{aligned} & (\tilde{\eta}(\mathbf{n}) + (-1)^{\epsilon(\eta)} i^{\tilde{l}} \delta(\mathbf{n} = \mathbf{o})) \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} \times \{ N(\mathbf{f}_\eta)^{-z} N(\mathbf{n})^{-z} L(z, \eta) \Upsilon_{S,l}^\eta(-z; \mathbf{s}) \\ & - N(\mathbf{f}_\eta)^z N(\mathbf{n})^z L(-z, \eta) \Upsilon_{S,l}^\eta(z; \mathbf{s}) \} dz \alpha(\mathbf{s}) d\mu_S(\mathbf{s}). \end{aligned}$$

As before, this gives an analytic continuation of  $\mathbb{W}_{\bar{\mathbf{u}}}^\eta(\beta, \lambda; \alpha)$ . We set  $f_{\bar{\mathbf{u}}}(z) = -N(\mathbf{f}_\eta)^{2z} N(\mathbf{n})^z f_{\mathbf{u}}(z)$ . Then,

$$\begin{aligned} & \operatorname{CT}_{\lambda=0} \mathbb{W}_{\bar{\mathbf{u}}}^\eta(\beta, \lambda; \alpha) \\ &= (\tilde{\eta}(\mathbf{n}) + (-1)^{\epsilon(\eta)} i^{\tilde{l}} \delta(\mathbf{n} = \mathbf{o})) (\operatorname{CT}_{z=0} \frac{f_{\bar{\mathbf{u}}}(z)}{z} \beta(0) + \frac{1}{2} \operatorname{Res}_{z=0} f_{\bar{\mathbf{u}}}(z) \beta''(0)). \end{aligned}$$

Since  $\eta$  is supposed to be nontrivial,  $f_{\bar{\mathbf{u}}}(z)$  is holomorphic at  $z = 0$  and

$$\begin{aligned} \operatorname{CT}_{z=0} \frac{f_{\bar{\mathbf{u}}}(z)}{z} &= f'_{\bar{\mathbf{u}}}(0) = -\log(N(\mathbf{n})N(\mathbf{f}_\eta^2)) f_{\mathbf{u}}(0) - f'_{\mathbf{u}}(0) \\ &= \mathcal{G}(\eta) D_F^{1/2} \{ -\pi^{\epsilon(\eta)} \Upsilon_S^\eta(\mathbf{s}) L(1, \eta) \log(N(\mathbf{n})N(\mathbf{f}_\eta^2)) - \mathfrak{W}_{S,\mathbf{u}}^\eta(l, \mathbf{n}|\mathbf{s}) \}. \end{aligned}$$

□

**Lemma 15.** *The integral  $\mathbb{W}_{\text{hyp}}^\eta(\beta, \lambda; \alpha)$  converges absolutely and has an analytic continuation to the region  $\operatorname{Re}(\lambda) > -\epsilon$  for some  $\epsilon > 0$ . The constant term of  $\mathbb{W}_{\text{hyp}}^\eta(\beta, \lambda; \alpha)$  at  $\lambda = 0$  equals  $\mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha)\beta(0)$ . Here*

$$\mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha) = \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \mathfrak{L}_\eta(l, \mathbf{n}|\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s})$$

with

$$\mathfrak{L}_\eta(l, \mathbf{n}|\mathbf{s}) = \sum_{b \in F - \{0, -1\}} \int_{\mathbb{A}^\times} \Psi_l^{(0)}(\mathbf{n}|\mathbf{s}, \delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_\eta \\ 0 & 1 \end{bmatrix}) \eta(tx_\eta^*) \log |t|_{\mathbb{A}} d^\times t.$$

*Proof.* The absolute convergence and analytic continuation of  $\mathbb{W}_{\text{hyp}}^\eta(\beta, \lambda; \alpha)$  are given in the same way as [7, Lemma 8.5]. We obtain the last assertion with the aid of (2.1).  $\square$

From the analysis so far, (3.1) yields the formula:

$$(3.2) \quad \partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha)) = \mathbb{W}_{\text{u}}^\eta(\beta, \lambda; \alpha) + \mathbb{W}_{\text{u}}^\eta(\beta, \lambda; \alpha) + \mathbb{W}_{\text{hyp}}^\eta(\beta, \lambda; \alpha)$$

which is valid on a half-plane  $\text{Re}(\lambda) > -\epsilon$  containing  $\lambda = 0$ .

**3.1. The relative trace formula.** For any ideal  $\mathfrak{m} \subset \mathfrak{o}$ , set

$$(3.3) \quad \iota(\mathfrak{m}) = [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{m})] = \prod_{v \in S(\mathfrak{m})} (1 + q_v) q_v^{\text{ord}_v(\mathfrak{m})-1}.$$

Let  $\mathcal{J}_{S, \eta}$  be the monoid of ideals generated by prime ideals  $\mathfrak{p}_v$  with  $v \in \Sigma_{\text{fin}} - S \cup S(\mathfrak{f}_\eta)$ . We shall introduce several functionals in  $\alpha \in \mathcal{A}_S$  depending on an ideal  $\mathfrak{m} \in \mathcal{J}_{S, \eta}$ :

$$(3.4) \quad \text{AL}^w(\mathfrak{m}; \alpha) = C_l \sum_{\pi \in \Pi_{\text{cus}}(l, \mathfrak{m})} \frac{w_{\mathfrak{m}}^\eta(\pi) \iota(\mathfrak{f}_\pi)}{N(\mathfrak{f}_\pi) \iota(\mathfrak{m})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi, \text{Ad})} \alpha(\nu_S(\pi)),$$

$$(3.5) \quad \text{AL}^{\partial w}(\mathfrak{m}; \alpha) = C_l \sum_{\pi \in \Pi_{\text{cus}}(l, \mathfrak{m})} \frac{\partial w_{\mathfrak{m}}^\eta(\pi) \iota(\mathfrak{f}_\pi)}{N(\mathfrak{f}_\pi) \iota(\mathfrak{m})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi, \text{Ad})} \alpha(\nu_S(\pi)),$$

$$(3.6) \quad \text{ADL}_\pm^w(\mathfrak{m}; \alpha) = C_l \sum_{\substack{\pi \in \Pi_{\text{cus}}(l, \mathfrak{m}) \\ \epsilon(1/2, \pi \otimes \eta) = \pm 1}} \frac{w_{\mathfrak{m}}^\eta(\pi) \iota(\mathfrak{f}_\pi)}{N(\mathfrak{f}_\pi) \iota(\mathfrak{m})} \frac{L(1/2, \pi) L'(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi, \text{Ad})} \alpha(\nu_S(\pi)).$$

The derivative of  $L$ -functions in  $\text{ADL}_+^w$  is eliminated by the functional equation.

**Proposition 16.** *We have*

$$\text{ADL}_+^w(\mathfrak{m}; \alpha) = C_l \sum_{\pi \in \Pi_{\text{cus}}(l, \mathfrak{m})} \log\{N(\mathfrak{f}_\pi \mathfrak{f}_\eta^2) D_F^2\}^{-1/2} \frac{w_{\mathfrak{m}}^\eta(\pi) \iota(\mathfrak{f}_\pi)}{N(\mathfrak{f}_\pi) \iota(\mathfrak{m})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi, \text{Ad})} \alpha(\nu_S(\pi)).$$

*Proof.* By the functional equation,

$$L'(1/2, \pi \otimes \eta) = \frac{\epsilon'(1/2, \pi \otimes \eta)}{2} L(1/2, \pi \otimes \eta)$$

if  $\epsilon(1/2, \pi \otimes \eta) = 1$ . An explicit form of the  $\epsilon$ -factor is given by  $\epsilon(s, \pi \otimes \eta) = \epsilon(1/2, \pi \otimes \eta) \{N(\mathfrak{f}_{\pi \otimes \eta}) D_F^2\}^{1/2-s} = \epsilon(1/2, \pi \otimes \eta) \{N(\mathfrak{f}_\pi) N(\mathfrak{f}_\eta)^2 D_F^2\}^{1/2-s}$ . Hence we obtain the assertion immediately.  $\square$

The following is the main consequence of this section.

**Theorem 17.** *For any ideal  $\mathbf{n} \in \mathcal{J}_{S, \eta}$  and for any  $\alpha \in \mathcal{A}_S$ ,*

$$(3.7) \quad 2^{-1}(-1)^{\#S+\epsilon(\eta)} \mathcal{G}(\eta) D_F^{-1} \{\text{ADL}_-^w(\mathbf{n}; \alpha) + \text{ADL}_+^w(\mathbf{n}; \alpha) + (\log D_F) \text{AL}^w(\mathbf{n}; \alpha) + \text{AL}^{\partial w}(\mathbf{n}; \alpha)\} \\ = \tilde{\mathbb{W}}_{\text{u}}^\eta(l, \mathbf{n}|\alpha) + \mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha).$$

Here

$$(3.8) \quad \tilde{W}_u^\eta(l, \mathbf{n}|\alpha) = (1 - (-1)^{\epsilon(\eta)} \tilde{\eta}(\mathbf{n})) (-1)^{\epsilon(\eta)} \mathcal{G}(\eta) D_F^{1/2} \{1 + (-1)^{\epsilon(\eta)} \tilde{\eta}(\mathbf{n}) i^{\tilde{l}} \delta(\mathbf{n} = \mathbf{o})\} \\ \times \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \tilde{\mathfrak{W}}_S^\eta(l, \mathbf{n}|\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s})$$

with  $d\mu_S(\mathbf{s}) = \prod_{v \in S} 2^{-1} \log q_v (q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2}) ds_v$  and  $\mathbb{L}_S(\mathbf{c})$  being the multidimensional contour  $\prod_{v \in S} \{\operatorname{Re}(s_v) = c_v\}$  directed usually,

(3.9)

$$\tilde{\mathfrak{W}}_S^\eta(l, \mathbf{n}|\mathbf{s}) = \pi^{\epsilon(\eta)} \Upsilon_S^\eta(\mathbf{s}) L(1, \eta) \left\{ \log(\sqrt{N(\mathbf{n})} D_F N(\mathbf{f}_\eta)) + \frac{L'(1, \eta)}{L(1, \eta)} + \mathfrak{C}(l) + \sum_{v \in S} \frac{\log q_v}{1 - \eta_v(\varpi_v) q_v^{(s_v+1)/2}} \right\}, \\ \Upsilon_S^\eta(\mathbf{s}) = \prod_{v \in S} (1 - \eta_v(\varpi_v) q_v^{-(1+s_v)/2})^{-1} (1 - q_v^{(1+s_v)/2})^{-1}, \\ \mathfrak{C}(l) = \sum_{v \in \Sigma_\infty} \left( \sum_{k=1}^{l_v/2-1} \frac{1}{k} - \frac{1}{2} \log \pi - \frac{1}{2} C_{\text{Euler}} - \delta_{\epsilon_v, 1} \log 2 \right).$$

*Proof.* From Proposition 8 together with [7, Lemma 6.4],

$$\text{CT}_{\lambda=0} \partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha)) \\ = 2^{-1} (-1)^{\#S + \epsilon(\eta)} \mathcal{G}(\eta) D_F^{-1} \{ \text{ADL}_-^w(\mathbf{n}; \alpha) + \text{ADL}_+^w(\mathbf{n}; \alpha) + (\log D_F) \text{AL}^w(\mathbf{n}; \alpha) + \text{AL}^{\partial w}(\mathbf{n}; \alpha) \}.$$

On the other hand, from the formula (3.2), the same  $\text{CT}_{\lambda=0} \partial P_{\beta, \lambda}^\eta(\hat{\Psi}_{\text{reg}}^l(\mathbf{n}|\alpha))$  is computed by Lemmas 12, 14 and 15.  $\square$

#### 4. EXTRACTION OF THE NEW PART : THE TOTALLY INERT CASE

Let  $\mathcal{I}_{S, \eta}$  be the monoid of ideals generated by prime ideals  $\mathfrak{p}_v$  such that  $v \in \Sigma_{\text{fin}} - S \cup S(\mathbf{f}_\eta)$  and  $\tilde{\eta}(\mathfrak{p}_v) = -1$ . Note that  $\mathcal{I}_{S, \eta}$  is a submonoid of  $\mathcal{J}_{S, \eta}$  defined in 3.1.

In this section, we separate the new part i.e., the contribution of those  $\pi$  with  $\mathbf{f}_\pi = \mathbf{n}$ , from the total average  $\text{ADL}_-^w(\mathbf{n}; \alpha)$  under the condition  $\mathbf{n} \in \mathcal{I}_{S, \eta}$ .

**4.1. The  $\mathcal{N}$ -transform.** For any ideal  $\mathbf{c}$  and a place  $v \in \Sigma_{\text{fin}}$ , set

$$\omega_v(\mathbf{c}) = \begin{cases} 1, & (v \in S(\mathbf{c})), \\ \frac{q_v + 1}{q_v - 1}, & (v \notin S(\mathbf{c})). \end{cases}$$

For any pair of integral ideals  $\mathbf{m}$  and  $\mathbf{b}$ , define

$$\omega(\mathbf{m}, \mathbf{b}) = \delta(\mathbf{m} \subset \mathbf{b}) \prod_{v \in S(\mathbf{b})} \omega_v(\mathbf{m} \mathbf{b}^{-1}).$$

Given an ideal  $\mathbf{n}$ , let  $\mathbf{n}_0$  denote the largest square free integral ideal dividing  $\mathbf{n}$ ; thus, there exists the unique integral ideal  $\mathbf{n}_1$  such that

$$\mathbf{n} = \mathbf{n}_0 \mathbf{n}_1^2.$$

Let  $\mathcal{I}$  be a set of integral ideals with the property that  $\mathbf{n} \subset \mathbf{m}$ ,  $\mathbf{n} \in \mathcal{I}$  implies  $\mathbf{m} \in \mathcal{I}$ .



**Proposition 18.** Let  $B(\mathfrak{m})$  and  $A(\mathfrak{m})$  be two arithmetic functions defined for ideals  $\mathfrak{m} \in \mathcal{I}$ . Then, the following two conditions are equivalent:

(i) For any  $\mathfrak{n} \in \mathcal{I}$ ,

$$B(\mathfrak{n}) = \sum_{\mathfrak{b}|\mathfrak{n}_1} \omega(\mathfrak{n}, \mathfrak{b}^2) A(\mathfrak{n}\mathfrak{b}^{-2}).$$

(ii) For any  $\mathfrak{n} \in \mathcal{I}$ ,

$$A(\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \left\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \right\} B(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}).$$

*Proof.* We show that (i) implies (ii). By substituting (i), the right-hand side of (ii) becomes

$$\begin{aligned} & \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \left\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \right\} \left\{ \sum_{\mathfrak{b}|\mathfrak{n}_1} \omega(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}, \mathfrak{b}^2) A(\mathfrak{n}\mathfrak{b}^{-2} \prod_{v \in I} \mathfrak{p}_v^{-2}) \right\} \\ &= \sum_{\mathfrak{b}_1|\mathfrak{n}_1} \left\{ \sum_{I \subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \omega \left( \mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}, \mathfrak{n}_1^2 \mathfrak{b}_1^{-2} \prod_{v \in I} \mathfrak{p}_v^{-2} \right) \prod_{v \in I \cap S_1(\mathfrak{n}_1) \cap S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} \omega_v(\mathfrak{n}_0) \right\} A(\mathfrak{n}_0 \mathfrak{b}_1^2) \end{aligned}$$

Here to have the equality, we made the substitution  $\mathfrak{b}_1 = \mathfrak{n}_1 \mathfrak{b}^{-1} \prod_{v \in I} \mathfrak{p}_v^{-1}$ . If  $\mathfrak{b}_1 = \mathfrak{n}_1$ , the term inside the bracket is 1 obviously; otherwise it equals

$$\begin{aligned} & \sum_{I \subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v \in S(\mathfrak{n}_1\mathfrak{b}_1^{-1} \prod_{v \in I} \mathfrak{p}_v^{-1}) - S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \prod_{v \in I \cap S(\mathfrak{n}_1\mathfrak{b}_1^{-1}) \cap S_1(\mathfrak{n}_1) - S(\mathfrak{n}_0)} \frac{q_v + 1}{q_v - 1} \\ &= \sum_{I \subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v \in [(I - S_1(\mathfrak{n}_1\mathfrak{b}_1^{-1})) \cup (S(\mathfrak{n}_1\mathfrak{b}_1^{-1}) - I)] - S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \prod_{v \in I \cap S_1(\mathfrak{n}_1\mathfrak{b}_1^{-1}) - S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \\ &= \sum_{I \subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v \in (S(\mathfrak{n}_1\mathfrak{b}_1^{-1}) - I) - S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \prod_{v \in I - S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \\ &= \prod_{v \in S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (\omega_v(\mathfrak{n}_0\mathfrak{b}_1^2) - \omega_v(\mathfrak{n}_0\mathfrak{b}_1^2)), \end{aligned}$$

which is zero by  $S(\mathfrak{n}_1\mathfrak{b}_1^{-1}) \neq \emptyset$ . This completes the proof.  $\square$

**Definition 19.** For an arithmetic function  $B : \mathcal{I} \rightarrow \mathbb{C}$ , we define its  $\mathcal{N}$ -transform  $\mathcal{N}[B] : \mathcal{I} \rightarrow \mathbb{C}$  by the formula

$$\mathcal{N}[B](\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \left\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \right\} \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} B(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}).$$

**Lemma 20.** For  $t \in \mathbb{C}$ , let  $N^t$  be the arithmetic function  $\mathfrak{n} \mapsto N(\mathfrak{n})^t$  on  $\mathcal{I}$ . For any ideal  $\mathfrak{n}$ , we have

$$\mathcal{N}[N^t](\mathfrak{n}) = N(\mathfrak{n})^t \left\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2(1+t)}) \right\} \left\{ \prod_{v \in S_2(\mathfrak{n})} (1 - (1 - q_v^{-1})^{-1} q_v^{-2(1+t)}) \right\}.$$

*Proof.* By (3.3), we have

$$\frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} = \prod_{v \in I} q_v^{-2} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 + q_v^{-1})^{-1}.$$

Therefore,

$$\begin{aligned} & \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \left\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \right\} \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} N(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})^t \\ &= N(\mathfrak{n})^t \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \left\{ \prod_{v \in I \cap S_2(\mathfrak{n})} \frac{q_v + 1}{q_v - 1} \right\} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 + q_v^{-1})^{-1} \left\{ \prod_{v \in I} q_v^{-2(1+t)} \right\} \\ &= N(\mathfrak{n})^t \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 - q_v^{-1})^{-1} \left\{ \prod_{v \in I} q_v^{-2t} \right\} \\ &= N(\mathfrak{n})^t \left\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2(1+t)}) \right\} \left\{ \prod_{v \in S_2(\mathfrak{n})} (1 - (1 - q_v^{-1})^{-1} q_v^{-2(1+t)}) \right\}. \end{aligned}$$

□

**Corollary 21.** *The  $\mathcal{N}$ -transform of the arithmetic function  $\log N(\mathfrak{n})$  on  $\mathcal{I}$  is given by*

$$\begin{aligned} \mathcal{N}[\log N](\mathfrak{n}) &= \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2}) \prod_{v \in S_2(\mathfrak{n})} (1 - (q_v^2 - q_v)^{-1}) \\ &\quad \times \left( \log N(\mathfrak{n}) + \sum_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} \frac{2 \log q_v}{q_v^2 - 1} + \sum_{v \in S_2(\mathfrak{n})} \frac{2 \log q_v}{q_v^2 - q_v - 1} \right). \end{aligned}$$

*Proof.* Take the derivative at  $t = 0$  of the formula in Lemma 20. □

For any arithmetic function  $B : \mathcal{I} \rightarrow \mathbb{C}$ , we define another function  $\mathcal{N}^+[B]$  by setting

$$\mathcal{N}^+[B](\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} \left\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \right\} \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} B(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})$$

for  $\mathfrak{n} = \mathfrak{n}_0 \mathfrak{n}_1^2 \in \mathcal{I}$ . In a similar way to Lemma 20, we have

$$(4.1) \quad \mathcal{N}^+[N^t] = N(\mathfrak{n})^t \left\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 + q_v^{-2(t+1)}) \right\} \left\{ \prod_{v \in S_2(\mathfrak{n})} (1 + (1 - q_v^{-1})^{-1} q_v^{-2(t+1)}) \right\}$$

for any  $t \in \mathbb{R}$ .

**Lemma 22.** *Let  $c > 0$ . Then for any sufficiently small  $\epsilon > 0$ , we have*

$$\mathcal{N}^+[N^{-c+\epsilon}](\mathfrak{n}) \ll_{\epsilon} N(\mathfrak{n})^{-\inf(c,1)+\epsilon}, \quad \mathfrak{n} \in \mathcal{I}.$$

*Proof.* From  $N(\mathfrak{n})^{-c+\epsilon} \leq N(\mathfrak{n})^{-\inf(c,1)+\epsilon}$ , we have  $\mathcal{N}^+[N^{-c+\epsilon}](\mathfrak{n}) \leq \mathcal{N}^+[N^{-\inf(c,1)+\epsilon}](\mathfrak{n})$  obviously. Let us set  $t = -\inf(c,1)+\epsilon$  and examine the right-hand side of the formula (4.1). We note that  $t+1 = 1 - \inf(c,1) + \epsilon \geq \epsilon > 0$ . The set  $P(\epsilon) = \{v \in \Sigma_{\text{fin}} \mid 1 - q_v^{-1} < q_v^{-\epsilon}\}$  is a finite set. For  $v \in S_2(\mathfrak{n}) - P(\epsilon)$ , we have  $(1 - q_v^{-1})^{-1} \leq q_v^{\epsilon}$  and  $q_v^{-2(t+1)} \leq q_v^{-2\epsilon}$ ; by

these, the factor  $1 + (1 - q_v^{-1})q_v^{-2(t+1)}$  is bounded by  $1 + q_v^{-\epsilon}$ . For  $v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})$  or  $v \in S_2(\mathfrak{n}) \cap P(\epsilon)$ , we simply apply  $q_v^{-2(t+1)} \leq q_v^{-2\epsilon}$ . Thus,

(4.2)

$$\mathcal{N}^+[N^t](\mathfrak{n}) \leq N(\mathfrak{n})^t \left\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 + q_v^{-2\epsilon}) \right\} \left\{ \prod_{v \in P(\epsilon)} (1 + (1 - q_v^{-1})^{-1} q_v^{-2\epsilon}) \right\} \left\{ \prod_{v \in S_2(\mathfrak{n}) - P(\epsilon)} (1 + q_v^{-\epsilon}) \right\}.$$

In the right-hand side, the second factor is independent of  $\mathfrak{n}$ . The first and the last factors combined are estimated as

$$\begin{aligned} \left\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 + q_v^{-2\epsilon}) \right\} \left\{ \prod_{v \in S_2(\mathfrak{n}) - P(\epsilon)} (1 + q_v^{-\epsilon}) \right\} &\leq \left\{ \prod_{v \in S(\mathfrak{n})} (1 + q_v^{-\epsilon}) \right\}^2 \\ &\ll_{\epsilon} \left\{ \prod_{v \in S(\mathfrak{n})} q_v^{\epsilon} \right\}^2 \leq N(\mathfrak{n})^{2\epsilon}. \end{aligned}$$

Hence there exists a constant  $C(\epsilon) > 0$  dependent of  $\epsilon$  such that (4.2) is less than  $C(\epsilon) N(\mathfrak{n})^{-\inf(c,1)+3\epsilon}$  for any  $\mathfrak{n} \in \mathcal{I}$ .  $\square$

**4.2. The totally inert case over  $\mathfrak{n}$ .** Set  $\mathcal{I} = \mathcal{I}_{S,\eta}$ . Fixing a test function  $\alpha \in \mathcal{A}_S$  for a while, we study the arithmetic functions  $\text{AL}^* : \mathcal{I} \rightarrow \mathbb{C}$  and  $\text{ADL}_-^* : \mathcal{I} \rightarrow \mathbb{C}$  defined by the formulas (1.1) and (1.3), respectively. We relate these functions to the  $\mathcal{N}$ -transforms of arithmetic functions  $\text{AL}^w, \text{ADL}_\pm^w$  on  $\mathcal{I}$  defined in 3.1.

We remark that an ideal  $\mathfrak{n} \in \mathcal{I}$  satisfies the condition

$$(4.3) \quad \eta_v(\varpi_v) = -1, \quad v \in S(\mathfrak{n}).$$

This means that the quadratic extension of  $F$  corresponding to  $\eta$  is inert over all places dividing  $\mathfrak{n}$ . Under this condition, the quantities  $w_{\mathfrak{n}}^{\eta}(\pi)$  and  $\partial w_{\mathfrak{n}}^{\eta}(\pi)$  turn out to be written explicitly in terms of the arithmetic function  $\omega(\mathfrak{m}, \mathfrak{b})$ .

**Lemma 23.** *Let  $\mathfrak{n} \in \mathcal{I}$ . Then, for any  $\pi \in \Pi_{\text{cus}}(l, \mathfrak{n})$ , we have  $w_{\mathfrak{n}}^{\eta}(\pi) = 0$  unless  $\mathfrak{n}\mathfrak{f}_{\pi}^{-1} = \mathfrak{b}^2$  for some integral ideal  $\mathfrak{b}$ , in which case*

$$w_{\mathfrak{n}}^{\eta}(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_{\pi}^{-1}).$$

*Proof.* Let  $v \in S(\mathfrak{n}\mathfrak{f}_{\pi}^{-1})$ . From [7, Lemma 6.2],

$$r(\pi_v, \eta_v) = \frac{1 + (-1)^{k_v}}{2} \times \begin{cases} 1, & (c(\pi_v) \geq 1), \\ \frac{q_v + 1}{q_v - 1}, & (c(\pi_v) = 0). \end{cases}$$

Thus  $r(\pi_v, \eta_v) = 0$  unless  $k_v = \text{ord}_v(\mathfrak{n}\mathfrak{f}_{\pi}^{-1})$  is even.  $\square$

**Lemma 24.** *Let  $\mathfrak{n} \in \mathcal{I}$ . For any  $\pi \in \Pi_{\text{cus}}(l, \mathfrak{n})$ , we have the following.*

(i) *If  $\mathfrak{n}\mathfrak{f}_{\pi}^{-1} = \mathfrak{b}^2$  with an integral ideal  $\mathfrak{b}$ , then*

$$\partial w_{\mathfrak{n}}^{\eta}(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_{\pi}^{-1}) \sum_{v \in S(\mathfrak{b})} (-\log q_v) \text{ord}_v(\mathfrak{b}).$$

(ii) If  $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2\mathfrak{p}_u$  with an integral ideal  $\mathfrak{b}$  and a place  $u \in S(\mathfrak{n})$ , then

$$\partial w_\pi^\eta(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_\pi^{-1}) \log q_u \begin{cases} \text{ord}_u(\mathfrak{b}) + \frac{q_u - 1}{(1 + a_u q_u^{1/2})(1 + a_u^{-1} q_u^{1/2})}, & (c(\pi_u) = 0), \\ \text{ord}_u(\mathfrak{b}) + \frac{1}{1 + q_u^{-1} \chi_u(\varpi_u)}, & (c(\pi_u) = 1), \\ \text{ord}_u(\mathfrak{b}) + 1, & (c(\pi_u) \geq 2). \end{cases}$$

Except the above two cases (i) and (ii), we have  $\partial w_\pi^\eta(\pi) = 0$ .

**Lemma 25.** For any  $\mathfrak{n} \in \mathcal{I}$ ,

$$\begin{aligned} \text{AL}^w(\mathfrak{n}; \alpha) &= \sum_{\mathfrak{b}} \omega(\mathfrak{n}, \mathfrak{b}^2) \frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})} \text{AL}^*(\mathfrak{n}\mathfrak{b}^{-2}; \alpha), \\ \text{ADL}_-^w(\mathfrak{n}; \alpha) &= \sum_{\mathfrak{b}} \omega(\mathfrak{n}, \mathfrak{b}^2) \frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})} \text{ADL}^*(\mathfrak{n}\mathfrak{b}^{-2}; \alpha), \\ \text{ADL}_+^w(\mathfrak{n}; \alpha) &= \sum_{\mathfrak{b}} \omega(\mathfrak{n}, \mathfrak{b}^2) \frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})} \log(N(\mathfrak{n}\mathfrak{b}^{-2})^{-1/2} N(\mathfrak{f}_\eta)^{-1} D_F^{-1}) \text{AL}^*(\mathfrak{n}\mathfrak{b}^{-2}; \alpha), \end{aligned}$$

where  $\mathfrak{b}$  runs through all the integral ideals such that  $\mathfrak{n} \subset \mathfrak{b}^2$ .

*Proof.* This follows immediately from Lemma 23. To have the last formula, we also need Proposition 16.  $\square$

**Lemma 26.** For any  $\mathfrak{n} \in \mathcal{I}$ ,

$$\begin{aligned} \text{AL}^*(\mathfrak{n}) &= \mathcal{N}[\text{AL}^w](\mathfrak{n}), \\ \text{ADL}^*(\mathfrak{n}) &= \mathcal{N}[\text{ADL}_-^w](\mathfrak{n}), \\ -\log(\sqrt{N(\mathfrak{n})} N(\mathfrak{f}_\eta) D_F) \text{AL}^*(\mathfrak{n}) &= \mathcal{N}[\text{ADL}_+^w](\mathfrak{n}) \end{aligned}$$

*Proof.* By Lemma 25, we obtain the first formula by applying Proposition 18 with  $B(\mathfrak{m}) = \iota(\mathfrak{m}) \text{AL}^w(\mathfrak{m}; \alpha)$  and  $A(\mathfrak{m}) = \iota(\mathfrak{m}) \text{AL}^*(\mathfrak{m}; \alpha)$  both defined for  $\mathfrak{m} \in \mathcal{I}$ . The remaining two formulas are proved in the same way.  $\square$

The formula (3.7) in Theorem 17 can be applied to an arbitrary ideal  $\mathfrak{m} \in \mathcal{I}$ . In the right-hand side of the formula, we have two terms  $\tilde{\mathbb{W}}_\mathfrak{u}^\eta(l, \mathfrak{m}|\alpha)$  and  $\mathbb{W}_{\text{hyp}}^\eta(l, \mathfrak{m}|\alpha)$ , which we regard as arithmetic functions in  $\mathfrak{m}$  for a while and consider their  $\mathcal{N}$ -transforms  $\mathcal{N}[\tilde{\mathbb{W}}_\mathfrak{u}^\eta]$  and  $\mathcal{N}[\mathbb{W}_{\text{hyp}}^\eta]$ . The following is the main result of this section.

**Proposition 27.** For any  $\mathfrak{n} \in \mathcal{I}$ , we have the identity among linear functionals in  $\alpha \in \mathcal{A}_S$ :

$$(4.4) \quad \begin{aligned} \text{ADL}^*(\mathfrak{n}) &= 2(-1)^{\#S+\epsilon(\eta)} \mathcal{G}(\eta)^{-1} D_F \{ \mathcal{N}[\tilde{\mathbb{W}}_\mathfrak{u}^\eta](\mathfrak{n}) + \mathcal{N}[\mathbb{W}_{\text{hyp}}^\eta](\mathfrak{n}) \} \\ &\quad + \log(N(\mathfrak{n})^{1/2} N(\mathfrak{f}_\eta)) \text{AL}^*(\mathfrak{n}) - \mathcal{N}[\text{AL}^{\partial w}](\mathfrak{n}). \end{aligned}$$

*Proof.* We take the  $\mathcal{N}$ -transform of both sides of the formula (3.7) regarding it as an identity among arithmetic functions on  $\mathcal{I}$ . Then apply Lemma 26.  $\square$

## 5. AN ERROR TERM ESTIMATE FOR AVERAGED $L$ -VALUES

In this section we prove the first asymptotic formula of Theorem 1. Recall the sets  $\mathcal{I}_{S,\eta}^\pm$ , to which  $\mathbf{n}$  should belong. We note that, by the sign of the functional equation,  $L(1/2, \pi)L(1/2, \pi \otimes \eta) = 0$  if  $\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})$  unless  $\mathbf{n} \in \mathcal{I}_{S,\eta}^+$ . Thus we restrict ourselves to those levels  $\mathbf{n}$  belonging to  $\mathcal{I}_{S,\eta}^+$ , for otherwise  $\text{AL}^*(\mathbf{n}; \alpha) = 0$ .

We have the following asymptotic result, whose proof is given in the next subsection.

**Proposition 28.** *Suppose  $\underline{l} = \inf_{v \in \Sigma_\infty} l_v \geq 6$ . For any ideal  $\mathbf{m} \in \mathcal{I}_{S(\mathbf{a}),\eta}^+$ , we have*

$$\text{AL}^w(\mathbf{m}; \alpha_{\mathbf{a}}) = 4D_F^{3/2} L_{\text{fin}}(1, \eta) N(\mathbf{a})^{-1/2} \delta_{\square}(\mathbf{a}_\eta^-) d_1(\mathbf{a}_\eta^+) + \mathcal{O}_{\epsilon, l, \eta}(N(\mathbf{a})^{c+2+\epsilon} N(\mathbf{m})^{-c+\epsilon})$$

for any ideal  $\mathbf{a}$  primes to  $\mathfrak{f}_\eta$ , where  $c = d_F^{-1}(\underline{l}/2 - 1)$ .

From this, we can deduce the asymptotic formula for the primitive part  $\text{AL}^*(\mathbf{n}; \alpha_{\mathbf{a}})$  stated in Theorem 1. Indeed, we apply the first formula of Lemma 26 substituting the expression of  $\text{AL}^w$  given in Proposition 28. The main and the error terms are computed by Lemmas 20 and 22, respectively. This completes the proof.

**5.1. The proof of Proposition 28.** For any place  $v \in \Sigma_{\text{fin}}$ , we define a function  $\Lambda_v : F_v - \{0, -1\} \rightarrow \mathbb{Z}$  by setting

$$\Lambda_v(b) = \delta(b \in \mathfrak{o}_v) \{\text{ord}_v(b(b+1)) + 1\}.$$

For an  $\mathfrak{o}$ -ideal  $\mathfrak{b}$ , we set

$$\tau^{S(\mathfrak{b})}(b) = \left\{ \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{b})} \Lambda_v(b) \right\} \prod_{v \in S(\mathfrak{b})} \delta(b \in \mathfrak{b}^{-1} \mathfrak{o}_v), \quad b \in F - \{0, -1\}.$$

For an even integer  $k (\geq 4)$  and a real valued character  $\varepsilon$  of  $\mathbb{R}^\times$ , let  $J^\varepsilon(k; b)$  ( $b \in \mathbb{R} - \{0, -1\}$ ) be the integral studied in [7, 10.3]; they are evaluated explicitly in [7, Lemma 10.15] as

$$J^1(k; b) = \begin{cases} (1+b)^{-k/2} \frac{2\Gamma(k/2)^2}{\Gamma(k)} {}_2F_1(k/2, k/2; k; (b+1)^{-1}), & (b(b+1) > 0), \\ 2 \log |(b+1)/b| P_{k/2-1}(2b+1) - \sum_{m=1}^{[k/4]} \frac{8(k-4m+1)}{(2m-1)(k-2m)} P_{k/2-2m}(2b+1), & (b(b+1) < 0), \end{cases}$$

$$J^{\text{sgn}}(k; b) = \begin{cases} 0, & (b(b+1) > 0), \\ 2\pi i P_{k/2-1}(2b+1), & (b(b+1) < 0), \end{cases}$$

where  $P_n(x)$  is the Legendre polynomial of degree  $n$ .

**Lemma 29.** *Let  $k$  be an even integer greater than 2 and  $\eta_v$  a real valued character of  $\mathbb{R}^\times$ . Then, for any  $\epsilon > 0$ , we have the estimation*

$$|b(b+1)|^\epsilon |J^{\eta_v}(k; b)| \ll_{\epsilon, k} (1+|b|)^{-k/2+2\epsilon}, \quad b \in \mathbb{R} - \{0, -1\}$$

with the implied constant depending on  $k$  and  $\epsilon$ .

*Proof.* For  $J^{\text{sgn}}(k; b)$  the estimation is obvious. As for  $J^1(k; b)$ , we only have to note that the estimation  ${}_2F_1(k/2, k/2; k; (b+1)^{-1}) = O(|\log b|)$  for small  $b > 0$  ([3, p.49]) and the functional equation  $J^1(k; b) = (-1)^{k/2} J^1(k; -b-1)$  ( $b < -1$ ) proved in [7, Lemma 10.16].  $\square$

Given relatively prime  $\mathfrak{o}$ -ideals  $\mathfrak{n}$  and  $\mathfrak{b}$  and for  $\epsilon \geq 0$ , we set

$$\mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{b}) = \sum_{b \in \mathfrak{n}\mathfrak{b}^{-1} - \{0, -1\}} \tau^{S(b)}(b)^2 |\mathbf{N}(b(b+1))|^\epsilon \prod_{v \in \Sigma_\infty} |J^{\eta_v}(l_v; b)|.$$

**Proposition 30.** *Suppose  $\underline{l} \geq 6$ . Let  $\mathfrak{b}$  and  $\mathfrak{n}$  be relatively prime ideals. For any  $\epsilon \geq 0$  and  $\epsilon' > 0$ , we have*

$$\mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{b}) \ll_{\epsilon, \epsilon', l} \mathbf{N}(\mathfrak{b})^{1+c+\epsilon'} \mathbf{N}(\mathfrak{n})^{-c+\epsilon+\epsilon'}$$

with the implied constant independent of  $\mathfrak{b}$  and  $\mathfrak{n}$ .

*Proof.* By [7, Lemma 12.4] and Lemma 29, we have

$$\mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{b}) \ll_{\epsilon, \epsilon', l} \mathbf{N}(\mathfrak{b})^{4\epsilon'} \sum_{b \in \mathfrak{n}\mathfrak{b}^{-1} - \{0\}} \prod_{v \in \Sigma_\infty} (1 + |b_v|)^{-l_v/2 + \epsilon + 2\epsilon'} = \mathbf{N}(\mathfrak{b})^{4\epsilon'} \theta(\mathfrak{n}\mathfrak{b}^{-1})$$

for any  $\epsilon \geq 0$  and any  $\epsilon' > 0$ , where we regard the fractional ideal  $\mathfrak{n}\mathfrak{b}^{-1}$  as a  $\mathbb{Z}$ -lattice in the Euclidean space  $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$  and  $\theta(\Lambda)$  is constructed for  $\{l_v - 2\epsilon - 4\epsilon'\}_{v \in \Sigma_\infty}$  in place of  $l$  (see §9). If  $\epsilon \geq 0$  and  $\epsilon' > 0$  are small enough, then we can apply the theory in §9 to this  $\theta(\Lambda)$ . The desired estimation follows if we apply Theorem 55 with  $\Lambda = \mathfrak{n}\mathfrak{b}^{-1}$  and  $\Lambda_0 = \mathfrak{b}^{-1}$  noting  $D(\mathfrak{n}\mathfrak{b}^{-1}) = \mathbf{N}(\mathfrak{n})\mathbf{N}(\mathfrak{b})^{-1}$ ,  $D(\mathfrak{b}^{-1}) = \mathbf{N}(\mathfrak{b})^{-1}$  and  $r(\mathfrak{b}^{-1}) \leq r(\mathfrak{o})$ .  $\square$

**Proposition 31.** *Suppose  $\underline{l} \geq 6$ . Given  $\mathfrak{o}$ -ideals  $\mathfrak{n}$  and  $\mathfrak{a} = \prod_{v \in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$  relatively prime to each other, for any  $\epsilon > 0$ , we have*

$$|\mathbb{J}_{\text{hyp}}^\eta(l, \mathfrak{n}|\alpha_{\mathfrak{a}})| \ll_{\epsilon, l, \eta} \mathbf{N}(\mathfrak{a})^{c+2+\epsilon} \mathbf{N}(\mathfrak{n})^{-c+\epsilon}$$

with the implied constant independent of  $\mathfrak{a}$  and  $\mathfrak{n}$ .

*Proof.* Let  $v \in S(\mathfrak{a})$  and  $n \in \mathbb{N}_0$ . By (1.4),

$$(5.1) \quad \alpha_{\mathfrak{p}_v^n}(\nu) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} = \sum_{m=0}^{[n/2]} \alpha_v^{(n-2m)}(\nu) - \delta(n \in 2\mathbb{N}_0)$$

with  $\alpha_v^{(m)}(\nu) = z^m + z^{-m}$ ,  $z = q_v^{\nu/2}$ . By [7, Lemma 10.3], we have

$$|J_v^{\eta_v}(b, \alpha_v^{(m)})| \ll (1+m)^2 \delta(|b|_v \leq q_v^m) q_v^{\delta(m>0)-m/2} \{1 + \Lambda_v(b)\}, \quad b \in F^\times - \{-1\}$$

with the implied constant independent of  $m \geq 0$  and  $v$ . Hence if  $n > 0$ ,

$$\begin{aligned} |J_v^{\eta_v}(b, \alpha_{\mathfrak{p}_v^n})| &\ll \delta(|b|_v \leq q_v^n) \left\{ \sum_{m=0}^n (1+m)^2 q_v^{1-m/2} \right\} \{1 + \Lambda_v(b)\} \\ &\leq \delta(|b|_v \leq q_v^n) q_v \left( \sum_{m=0}^{\infty} (1+m)^2 2^{-m/2} \right) \{1 + \Lambda_v(b)\}. \end{aligned}$$

Thus we have a constant  $C$  independent of  $v \in S(\mathfrak{a})$  and  $n \in \mathbb{N}_0$  such that

$$(5.2) \quad |J_v^{\eta_v}(b, \alpha_{\mathfrak{p}_v^n})| \leq C q_v^{\delta(n>0)} \delta(|b|_v \leq q_v^n) \{1 + \Lambda_v(b)\}, \quad b \in F^\times - \{0, -1\}.$$

Combining (5.2) with Proposition 30 and [7, Lemmas 10.4, 10.5 and Corollary 10.11], we obtain

$$\begin{aligned}
|\mathbb{J}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha_{\mathbf{a}})| &\leq C^{\#S(\mathbf{a})} \left\{ \prod_{v \in S(\mathbf{a})} q_v^{\delta(n_v > 0)} \right\} \sum_{I \subset S(\mathbf{a})} \sum_{b \in \mathbf{n}(\prod_{v \in I} \mathfrak{p}_v^{n_v})^{-1} \mathfrak{f}_\eta^{-1}} \tau^{S(\prod_{v \in I} \mathfrak{p}_v^{n_v})}(b) \prod_{v \in \Sigma_\infty} |J_v^{\eta_v}(l_v; b)| \\
&\leq C^{\#S(\mathbf{a})} N(\mathbf{a}) \sum_{I \subset S(\mathbf{a})} \mathfrak{I}_0(l, \mathbf{n}, \prod_{v \in I} \mathfrak{p}_v^{n_v} \mathfrak{f}_\eta) \\
&\ll_{\epsilon, l} C^{\#S(\mathbf{a})} N(\mathbf{a}) \sum_{I \subset S(\mathbf{a})} N(\prod_{v \in I} \mathfrak{p}_v^{n_v} \mathfrak{f}_\eta)^{1+c+\epsilon} N(\mathbf{n})^{-c+\epsilon} \\
&\ll_{\epsilon, l, \eta} C^{\#S(\mathbf{a})} N(\mathbf{a}) \times 2^{\#S(\mathbf{a})} N(\mathbf{a})^{1+c+\epsilon} N(\mathbf{n})^{-c+\epsilon}.
\end{aligned}$$

By the estimate  $(2C)^{\#S(\mathbf{a})} \ll_{\epsilon, \eta} N(\mathbf{a})^\epsilon$ , we are done.  $\square$

**Lemma 32.** Set  $\Upsilon_v^{\eta_v}(s) = (1 - \eta_v(\varpi_v) q_v^{-(1+s)/2})^{-1} (1 - q_v^{(1+s)/2})^{-1}$ . For  $n \in \mathbb{N}_0$ ,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Upsilon_v^{\eta_v}(s) \alpha_{\mathfrak{p}_v^n}(s) d\mu_v(s) &= -q_v^{-n/2} \begin{cases} \delta(n \in 2\mathbb{N}_0), & (\eta_v(\varpi_v) = -1), \\ n+1, & (\eta_v(\varpi_v) = +1), \end{cases} \\
\frac{\log q_v}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Upsilon_v^{\eta_v}(s) \alpha_{\mathfrak{p}_v^n}(s)}{1 - \eta_v(\varpi_v) q_v^{(s+1)/2}} d\mu_v(s) &= q_v^{-n/2} \log q_v \begin{cases} (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor, & (\eta_v(\varpi_v) = -1), \\ \frac{n(n+1)}{2}, & (\eta_v(\varpi_v) = +1). \end{cases}
\end{aligned}$$

*Proof.* The second integral is  $\tilde{U}_v^{\eta_v}(\alpha_{\mathfrak{p}_v^n})$  defined by (8.11). Then we have the second formula using (5.1) and Lemma 54 by a direct computation. The first formula is confirmed in the same way by using [7, Proposition 11.1].  $\square$

To show Proposition 28, we apply [7, Theorem 9.1] taking  $S = S(\mathbf{a})$ . From the first formula of Lemma 32,

$$\begin{aligned}
\mathbb{J}_{\mathbf{u}}^\eta(l, \mathbf{n}|\alpha_{\mathbf{a}}) &= (-1)^{\#S(\mathbf{a})} \prod_{v \in S(\mathbf{a}_\eta^-)} q_v^{-n_v/2} \delta(n_v \in 2\mathbb{N}_0) \prod_{v \in S(\mathbf{a}_\eta^+)} q_v^{-n_v/2} (n_v + 1) \\
&= (-1)^{\#S(\mathbf{a})} N(\mathbf{a})^{-1/2} \delta_{\square}(\mathbf{a}_\eta^-) d_1(\mathbf{a}_\eta^+).
\end{aligned}$$

We use Proposition 31 to estimate  $\mathbb{J}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha_{\mathbf{a}})$ , which yields the error term. This completes the proof.

## 6. AN ERROR TERM ESTIMATE FOR AVERAGED DERIVATIVE OF $L$ -VALUES

Let  $\mathbf{a} = \prod_{v \in S(\mathbf{a})} \mathfrak{p}_v^{n_v}$  and  $\mathcal{I}_{S, \eta}^\pm$  be as in §1. In this section we prove the asymptotic formula of  $\text{ADL}_-^*(\mathbf{n}; \alpha_{\mathbf{a}})$  for  $\mathbf{n} \in \mathcal{I}_{S(\mathbf{a}), \eta}^-$  stated in Theorem 1. We remark that  $\text{ADL}_-^*(\mathbf{n}) = 0$  if  $\mathbf{n} \in \mathcal{I}_{S(\mathbf{a}), \eta}^+$ . Indeed, for such  $\mathbf{n}$ ,  $\epsilon(1/2, \pi)\epsilon(1/2, \pi \otimes \eta) = +1$  for all  $\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})$ , which means  $\epsilon(1/2, \pi) = -1$  and hence  $L(1/2, \pi) = 0$  for all  $\pi$  occurring in the sum  $\text{ADL}_-^*(\mathbf{n})$ .

Starting from the formula (4.4) with  $\alpha$  specialized to  $\alpha_{\mathbf{a}}$ , we examine the 4 terms in the right-hand side separately. Here is the highlights in the analysis for each term.

- (i) We compute the term  $\mathcal{N}[\tilde{\mathbb{W}}_{\mathbf{u}}^\eta](\mathbf{n})$  explicitly by using Lemma 54, Lemma 20 and Corollary 21, which yields the main term of the formula (modulo a part of the error term); see 6.1 for detail.



(ii) We prove

$$\mathcal{N}[\mathbb{W}_{\text{hyp}}^\eta](\mathbf{n}) = \mathcal{O}_\epsilon(N(\mathbf{a})^{c+2+\epsilon}N(\mathbf{n})^{-\inf(1,c)+\epsilon})$$

by using the explicit formula of local terms given in §8; see 6.2 for detail.

(iii) Since  $\mathbf{n} \in \mathcal{I}_{S(\mathbf{a}),\eta}^-$ , the term  $\text{AL}^*(\mathbf{n})$  vanishes by the reason of the sign of the functional equations.

(iv) We prove

$$\mathcal{N}[\text{AL}^{\partial w}](\mathbf{n}) = \mathcal{O}_\epsilon(N(\mathbf{a})^{-1/2+\epsilon}X(\mathbf{n}) + N(\mathbf{a})^{c+2}N(\mathbf{n})^{-\inf(1,c)+\epsilon}).$$

This part is most subtle and the term  $X(\mathbf{n})$  arises from this stage; see 6.3 for detail.

Combining these considerations, we obtain the second formula in Theorem 1 immediately.

**6.1. Computation of  $\mathcal{N}[\tilde{\mathbb{W}}_\mathbf{u}^\eta](\mathbf{n})$ .** Let us describe the procedure (i). We take  $\alpha$  to be the function  $\alpha_\mathbf{a}$ . Set  $S = S(\mathbf{a})$ . From (3.8), we have that  $\mathcal{N}[\tilde{\mathbb{W}}_\mathbf{u}^\eta](\mathbf{n})$  is the sum of the following two integrals:

$$(6.1) \quad 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathcal{N}[\tilde{\mathfrak{W}}_S(-|\mathbf{s})](\mathbf{n})\alpha_\mathbf{a}(\mathbf{s})d\mu_S(\mathbf{s}),$$

$$(6.2) \quad 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathcal{N}[D\tilde{\mathfrak{W}}_S(-|\mathbf{s})](\mathbf{n})\alpha_\mathbf{a}(\mathbf{s})d\mu_S(\mathbf{s}),$$

where  $\tilde{\mathfrak{W}}_S(-|\mathbf{s})$  is the quantity (3.9) viewed as an arithmetic function in  $\mathbf{n}$  and  $D$  is an arithmetic function given by  $D(\mathbf{n}) = (-1)^{\epsilon(\eta)}\tilde{\eta}(\mathbf{n})\delta(\mathbf{n} = \mathbf{o})i^{\tilde{l}}$ . By the formula (3.9),

$$\begin{aligned} \mathcal{N}[\tilde{\mathfrak{W}}_S(-|\mathbf{s})](\mathbf{n}) &= \pi^{\epsilon(\eta)}\Upsilon_S^\eta(\mathbf{s})L(1,\eta)\left\{2^{-1}\mathcal{N}[\log N](\mathbf{n})\right. \\ &\quad \left. + \left(\log(D_F N(\mathbf{f}_\eta)) + \frac{L'}{L}(1,\eta) + \mathfrak{C}(l) + \sum_{v \in S} \frac{\log q_v}{1 - \eta_v(\varpi_v)q_v^{(s_v+1)/2}}\right)\mathcal{N}[1](\mathbf{n})\right\}. \end{aligned}$$

By Lemma 20 and Corollary 21, we have formulas of  $\mathcal{N}[\log N](\mathbf{n})$  and of  $\mathcal{N}[1](\mathbf{n})$ ; substituting these, and by using Lemma 32, we complete the evaluation of the integral (6.1).

The evaluation of the integral (6.2) is similar; instead of  $\mathcal{N}[\log N]$  and  $\mathcal{N}[1]$ , we need  $\mathcal{N}[D \log N]$  and  $\mathcal{N}[D]$ , which are much easier. Indeed, in the expression

$$\begin{aligned} \mathcal{N}[D \log N](\mathbf{n}) &= (-1)^{\epsilon(\eta)}i^{\tilde{l}}\sum_{I \subset S(\mathbf{a})}(-1)^{\#I}\left\{\prod_{v \in I \cap S_1(\mathbf{a})}\omega_v(\mathbf{n}_0)\right\}\frac{\iota(\mathbf{n}\prod_{v \in I}\mathfrak{p}_v^{-2})}{\iota(\mathbf{n})} \\ &\quad \times \tilde{\eta}(\mathbf{n}\prod_{v \in I}\mathfrak{p}_v^{-2})\delta(\mathbf{n}\prod_{v \in I}\mathfrak{p}_v^{-2} = \mathbf{o})\log N(\mathbf{n}\prod_{v \in I}\mathfrak{p}_v^{-2}), \end{aligned}$$

the sum survives only if  $\mathfrak{n} = \prod_{v \in S(\mathfrak{n})} \mathfrak{p}_v^2$  and  $I = S(\mathfrak{n})$ . A similar remark is applied to  $\mathcal{N}[D](\mathfrak{n})$ . Hence,

$$\begin{aligned}\mathcal{N}[D \log N](\mathfrak{n}) &= \delta(S(\mathfrak{n}) = S_2(\mathfrak{n})) \left\{ \prod_{v \in S(\mathfrak{n})} \frac{q_v + 1}{q_v - 1} \right\} (-1)^{\epsilon(\eta)} i^{\tilde{l}} \frac{(-1)^{\#S(\mathfrak{n})}}{\iota(\mathfrak{n})} \log N(\mathfrak{o}) = 0, \\ \mathcal{N}[D](\mathfrak{n}) &= \delta(S(\mathfrak{n}) = S_2(\mathfrak{n})) \left\{ \prod_{v \in S(\mathfrak{n})} \frac{q_v + 1}{q_v - 1} \right\} (-1)^{\epsilon(\eta)} i^{\tilde{l}} \frac{(-1)^{\#S(\mathfrak{n})}}{\iota(\mathfrak{n})}.\end{aligned}$$

Since  $\iota(\mathfrak{n})^{-1} = \mathcal{O}(N(\mathfrak{n})^{-2})$ , the integral (6.2) amounts at most to  $N(\mathfrak{n})^{-2+\epsilon} N(\mathfrak{a})^{-1/2+\epsilon}$ .

**6.2. Estimation of the term  $\mathcal{N}[\mathbb{W}_{\text{hyp}}^\eta](\mathfrak{n})$ .** Let us describe the procedure (ii). We need the following estimation, which we prove in 8.4.

**Proposition 33.** *For any small  $\epsilon > 0$ ,*

$$|\mathbb{W}_{\text{hyp}}^\eta(l, \mathfrak{n}; \alpha_{\mathfrak{a}})| \ll_{\epsilon, l, \eta} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-c+\epsilon}, \quad \mathfrak{n} \in \mathcal{I}_{S(\mathfrak{a}), \eta}^-$$

where the implied constant is independent of the ideal  $\mathfrak{a}$ .

From this proposition and Lemma 22,

$$|\mathcal{N}[\mathbb{W}_{\text{hyp}}^\eta](\mathfrak{n})| \leq \mathcal{N}^+[\mathbb{W}_{\text{hyp}}^\eta](\mathfrak{n}) \ll_{\epsilon} N(\mathfrak{a})^{c+2+\epsilon} \mathcal{N}^+[N^{-c+\epsilon}](\mathfrak{n}) \ll_{\epsilon} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-\inf(c, 1)+2\epsilon}.$$

**6.3. Estimation of the term  $\mathcal{N}[\text{AL}^{\partial w}](\mathfrak{n})$ .** Let us describe the procedure (iv).

**Lemma 34.** *Let  $\alpha \in \mathcal{A}_S$ . Then for any  $\mathfrak{n} \in \mathcal{I}_{S(\mathfrak{a}), \eta}^-$ , we have the inequality*

$$|\text{AL}^{\partial w}(\mathfrak{n}; \alpha)| \leq \sum_{(\mathfrak{b}, u)} D(\mathfrak{n}; \mathfrak{b}, u) \frac{\iota(\mathfrak{n} \mathfrak{b}^{-2} \mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})} |\text{AL}^*(\mathfrak{n} \mathfrak{b}^{-2} \mathfrak{p}_u^{-1}; \alpha)|,$$

where  $(\mathfrak{b}, u)$  runs through all the pairs of an integral ideal  $\mathfrak{b}$  and a place  $u$  such that  $\mathfrak{n} \subset \mathfrak{b}^2 \mathfrak{p}_u$ . For such  $(\mathfrak{b}, u)$ , we set

$$D(\mathfrak{n}; \mathfrak{b}, u) = \omega(\mathfrak{n}, \mathfrak{b}^2 \mathfrak{p}_u) (\log q_u) \left( \text{ord}_u(\mathfrak{b}) + \frac{q_u^{1/2} + 1}{q_u^{1/2} - 1} \right).$$

*Proof.* By Lemma 24, the  $\pi$ -summand of  $\text{AL}^{\partial w}(\mathfrak{n}; \alpha)$  vanishes unless the conductor  $\mathfrak{f}_\pi$  satisfies either (i)  $\mathfrak{n} \mathfrak{f}_\pi^{-1} = \mathfrak{b}^2$  with some  $\mathfrak{n} \subset \mathfrak{b}$ , or (ii)  $\mathfrak{n} \mathfrak{f}_\pi^{-1} = \mathfrak{b}^2 \mathfrak{p}_u$  with some  $\mathfrak{n} \subset \mathfrak{b}$  and  $u \in S(\mathfrak{n})$ . In the case (i), the  $\pi$ -summand vanishes. Indeed,  $\mathfrak{f}_\pi$  belongs to  $\mathcal{I}_{S(\mathfrak{a}), \eta}^-$  and thus  $L(1/2, \pi) L(1/2, \pi \otimes \eta) = 0$  by the functional equation. In the second case (ii), by the Ramanujan bound  $|a_v| = 1$  and the obvious relation  $|\chi_v(\varpi_v)| = 1$ , we have

$$\begin{aligned}|\partial w_{\mathfrak{n}}^\eta(\pi)| &\leq \omega(\mathfrak{n}, \mathfrak{b}^2 \mathfrak{p}_u) \log q_u \begin{cases} \text{ord}_u(\mathfrak{b}) + \frac{q_u - 1}{(1 - q_u^{1/2})^2}, & (c(\pi_u) = 0), \\ \text{ord}_u(\mathfrak{b}) + \frac{1}{1 - q_u^{-1}}, & (c(\pi_u) = 1), \\ \text{ord}_u(\mathfrak{b}) + 1, & (c(\pi_u) \geq 2) \end{cases} \\ &\leq \omega(\mathfrak{n}, \mathfrak{b}^2 \mathfrak{p}_u) (\log q_u) \left( \frac{q_u^{1/2} + 1}{q_u^{1/2} - 1} + \text{ord}_v(\mathfrak{b}) \right) = D(\mathfrak{n}; \mathfrak{b}, u).\end{aligned}$$

Here, we used  $\frac{1}{1 - q_u^{-1}} < \frac{q_u - 1}{(1 - q_u^{1/2})^2} = \frac{q_u^{1/2} + 1}{q_u^{1/2} - 1}$  to have the second inequality. □

**Lemma 35.** *For any small  $\epsilon \in (0, 1)$ , we have*

$$(6.3) \quad \sum_{(\mathbf{b}, u)} N(\mathbf{b}^2 \mathbf{p}_u)^\epsilon \frac{\iota(\mathbf{n} \mathbf{b}^{-2} \mathbf{p}_u^{-1})}{\iota(\mathbf{n})} N(\mathbf{n} \mathbf{b}^{-2} \mathbf{p}_u^{-1})^{-\inf(c, 1) + \epsilon} \ll_\epsilon N(\mathbf{n})^{-\inf(c, 1) + 2\epsilon},$$

$$(6.4) \quad \sum_{(\mathbf{b}, u)} N(\mathbf{b})^\epsilon \left( \frac{q_u + 1}{q_u - 1} \right)^2 (\log q_u) \frac{\iota(\mathbf{n} \mathbf{b}^{-2} \mathbf{p}_u^{-1})}{\iota(\mathbf{n})} \ll_\epsilon X(\mathbf{n}),$$

where  $(\mathbf{b}, u)$  runs through the same range as in Lemma 34.

*Proof.* Let us show the second estimate. By the inequality  $\iota(\mathbf{n} \mathbf{b}^{-2} \mathbf{p}_u^{-1}) / \iota(\mathbf{n}) \leq N(\mathbf{b}^{-2} \mathbf{p}_u^{-1})$ ,

$$\begin{aligned} \sum_{(\mathbf{b}, u)} N(\mathbf{b})^\epsilon \left( \frac{q_u + 1}{q_u - 1} \right)^2 (\log q_u) \frac{\iota(\mathbf{n} \mathbf{b}^{-2} \mathbf{p}_u^{-1})}{\iota(\mathbf{n})} &\leq \sum_{(\mathbf{b}, u)} N(\mathbf{b})^{-2+\epsilon} \left( \frac{q_u + 1}{q_u - 1} \right)^2 \frac{\log q_u}{q_u} \\ &\leq \left\{ \sum_{\mathbf{b} \subset \mathbf{o}} N(\mathbf{b})^{-2+\epsilon} \right\} \left\{ \sum_{u \in S(\mathbf{n})} \left( \frac{q_u + 1}{q_u - 1} \right)^2 \frac{\log q_u}{q_u} \right\} \\ &= \zeta_{F, \text{fin}}(2 - \epsilon) \left\{ \sum_{u \in S(\mathbf{n})} \frac{\log q_u}{q_u} + \sum_{u \in S(\mathbf{n})} \frac{4 \log q_u}{(q_u - 1)^2} \right\}. \end{aligned}$$

Since  $\zeta_{F, \text{fin}}(2 - \epsilon)$  is convergent, we are done.  $\square$

**Proposition 36.** *For any sufficiently small  $\epsilon > 0$ ,*

$$|\text{AL}^{\partial w}(\mathbf{n}; \alpha_a)| \ll_{\epsilon, l, \eta} N(\mathbf{a})^{-1/2} d_1(\mathbf{a}_\eta^+) \delta_\square(\mathbf{a}_\eta^-) X(\mathbf{n}) + N(\mathbf{a})^{c+2+\epsilon} N(\mathbf{n})^{-\inf(c, 1) + \epsilon}, \quad \mathbf{n} \in \mathcal{I}_{S(\mathbf{a}), \eta}^-.$$

*Proof.* Let  $\epsilon > 0$ . From  $\frac{x+1}{x-1} \ll_\epsilon x^\epsilon$  for  $x \geq 2$ , we have

$$\omega(\mathbf{n}, \mathbf{b}^2 \mathbf{p}_u) \leq \left( \prod_{v \in S(\mathbf{b})} \frac{q_v + 1}{q_v - 1} \right) \frac{q_u + 1}{q_u - 1} \ll_\epsilon N(\mathbf{b})^\epsilon \frac{q_u + 1}{q_u - 1}$$

with the implied constant independent of  $\mathbf{n}$  and  $(\mathbf{b}, u)$ . By this,

$$D(\mathbf{n}; \mathbf{b}, u) \ll_\epsilon N(\mathbf{b})^\epsilon (\log q_u) \left( \frac{q_u + 1}{q_u - 1} \right)^2$$

with the implied constant independent of  $\mathbf{n}$  and  $(\mathbf{b}, u)$ . Using these estimates, we have the desired bound by (1.6) and Lemmas 34 and 35.  $\square$

**Proposition 37.** *For any sufficiently small  $\epsilon > 0$ ,*

$$(6.5) \quad |\mathcal{N}[\text{AL}^{\partial w}](\mathbf{n}; \alpha_a)| \ll_{\epsilon, l, \eta} N(\mathbf{a})^{-1/2} d_1(\mathbf{a}_\eta^+) \delta_\square(\mathbf{a}_\eta^-) X(\mathbf{n}) + N(\mathbf{a})^{c+2+\epsilon} N(\mathbf{n})^{-\inf(1, c) + \epsilon}, \quad \mathbf{n} \in \mathcal{I}_{S(\mathbf{a}), \eta}^-.$$

*Proof.* Let  $\mathbf{n} = \mathbf{n}_1^2 \mathbf{n}_0$ . From Proposition 36, we have

$$|\mathcal{N}[\text{AL}^{\partial w}](\mathbf{n}; \alpha_a)| \ll_\epsilon N(\mathbf{a})^{-1/2} d_1(\mathbf{a}_\eta^+) \delta_\square(\mathbf{a}_\eta^-) \mathcal{N}^+[X](\mathbf{n}) + N(\mathbf{a})^{c+2+\epsilon} \mathcal{N}^+[N^{-\inf(1, c) + \epsilon}](\mathbf{n})$$

for all  $\mathfrak{n}$ . Since  $X(\mathfrak{m}) \leq X(\mathfrak{n})$  if  $\mathfrak{n} \subset \mathfrak{m} \subset \mathfrak{o}$ , we have

$$\begin{aligned} \mathcal{N}^+[X](\mathfrak{n}) &\leq X(\mathfrak{n}) \mathcal{N}^+[1](\mathfrak{n}) \\ &= X(\mathfrak{n}) \left\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 + q_v^{-2}) \right\} \left\{ \prod_{v \in S_2(\mathfrak{n})} (1 + (1 - q_v^{-1})^{-1} q_v^{-2}) \right\} \\ &\leq X(\mathfrak{n}) \left\{ \prod_{v \in \Sigma_{\text{fin}}} (1 + q_v^{-2}) \right\} \left\{ \prod_{v \in \Sigma_{\text{fin}}} (1 + (1 - q_v^{-1})^{-1} q_v^{-2}) \right\} \ll X(\mathfrak{n}), \end{aligned}$$

because the Euler products occurring are convergent.

From the proof of Lemma 22, we have  $\mathcal{N}^+[N^{-\inf(c,1)+\epsilon}](\mathfrak{n}) \ll_{\epsilon} N(\mathfrak{n})^{-\inf(c,1)+3\epsilon}$ . Consequently, for any sufficiently small  $\epsilon \in (0, 1)$ , we obtain the estimate

$$|\mathcal{N}[\text{AL}^{\partial w}](\mathfrak{n}; \alpha_{\mathfrak{a}})| \ll_{\epsilon} N(\mathfrak{a})^{-1/2} d_1(\mathfrak{a}_{\eta}^+) \delta_{\square}(\mathfrak{a}_{\eta}^-) X(\mathfrak{n}) + N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-\inf(1,c)+3\epsilon}$$

with the implied constant independent of  $\mathfrak{n}$  and  $\mathfrak{a}$ . Since  $\epsilon$  is arbitrary, we are done.  $\square$

## 7. AN ESTIMATION OF NUMBER OF CUSP FORMS

Recall that we set  $c = d_F^{-1}(\underline{l}/2 - 1)$ . Suppose that for each ideal  $\mathfrak{a} \subset \mathfrak{o}$ , we are given a set  $\mathcal{J}_{\mathfrak{a}}$  consisting of ideals prime to  $\mathfrak{f}_{\eta}\mathfrak{a}$  in such a way that  $\mathcal{J}_{\mathfrak{a}} \subset \mathcal{J}_{\mathfrak{a}'}$  for any  $\mathfrak{a} \subset \mathfrak{a}'$ , and a family of real numbers  $\{\omega_{\mathfrak{n}}(\pi) \mid \pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n})\}$  for each  $\mathfrak{n} \in \mathcal{J}_{\mathfrak{a}}$  which satisfies the following estimate for any  $\epsilon > 0$ :

$$(7.1) \quad \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n})} \omega_{\mathfrak{n}}(\pi) \prod_{v \in S(\mathfrak{a})} X_{n_v}(\lambda_v(\pi)) - \prod_{v \in S(\mathfrak{a})} \mu_{v, \eta_v}(X_{n_v}) \right| \ll_{\epsilon, l, \eta} \frac{N(\mathfrak{a})^{-1/2+\epsilon}}{\log N(\mathfrak{n})} + N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-\inf(c,1)+\epsilon},$$

with the implied constant independent of  $\mathfrak{a}$  and  $\mathfrak{n} \in \mathcal{J}_{\mathfrak{a}}$ . Moreover we impose the non-negativity condition:

$$(7.2) \quad \omega_{\mathfrak{n}}(\pi) \geq 0 \quad \text{for all } \pi \in \Pi_{\text{cus}}^*(l, \mathfrak{n}) \text{ and } \mathfrak{n} \in \mathcal{J}_{\mathfrak{a}}.$$

Let  $\mathfrak{q}$  be a prime ideal relatively prime to  $\mathfrak{f}_{\eta}$ . In what follows, we abuse the symbol  $\mathfrak{q}$  to denote the corresponding place  $v_{\mathfrak{q}}$  of  $F$ ; for example, we write  $\nu_{\mathfrak{q}}(\pi)$ ,  $\lambda_{\mathfrak{q}}(\pi)$  in place of  $\nu_{v_{\mathfrak{q}}}(\pi)$ ,  $\lambda_{v_{\mathfrak{q}}}(\pi)$  etc. Let  $S = \{v_1, \dots, v_r\}$  be a finite subset of  $\Sigma_{\text{fin}} - S(\mathfrak{f}_{\eta}\mathfrak{q})$  and set  $\mathfrak{a}_S = \prod_{v \in S} \mathfrak{p}_v$ . Let  $\mathbf{J} = \{J_j\}_{j=1}^r$  a family of closed subintervals of  $(-2, 2)$ . For each  $J_j$ , we choose an open interval  $J'_j$  such that  $\overline{J'_j} \subset J_j^{\circ}$  and  $C^{\infty}$ -function  $\chi_j : \mathbb{R} \rightarrow [0, \infty)$  with the following properties:

- $\chi_j(x) \neq 0$  for all  $x \in J'_j$ .
- $\text{supp}(\chi_j) \subset J_j$ .
- $\int_{-2}^2 \chi_j(x) d\mu_{v, \eta_v}(x) = 1$ , where

$$d\mu_{v, \eta_v}(x) = \begin{cases} \frac{q_v - 1}{(q_v^{1/2} + q_v^{-1/2} - x)^2} d\mu^{\text{ST}}(x), & (\eta_v(\varpi_v) = +1), \\ \frac{q_v + 1}{(q_v^{1/2} + q_v^{-1/2})^2 - x^2} d\mu^{\text{ST}}(x), & (\eta_v(\varpi_v) = -1). \end{cases}$$

Here  $d\mu^{\text{ST}}(x) = (2\pi)^{-1}\sqrt{4-x^2}dx$ . Fixing such a family of functions  $\{\chi_j\}$ , we set

$$\Omega_{\mathbf{n}}(\pi) = \omega_{\mathbf{n}}(\pi) \prod_{j=1}^r \chi_j(\lambda_{v_j}(\pi)), \quad \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \quad \mathbf{n} \in \mathcal{J}_{\mathbf{qa}_S}.$$

**Lemma 38.** *For any sufficiently small  $\epsilon > 0$ , there exists  $N_{\epsilon, S, l} > 0$  such that*

$$(7.3) \quad \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \Omega_{\mathbf{n}}(\pi) X_n(\lambda_{\mathbf{q}}(\pi)) - \mu_{\mathbf{q}, \eta_{\mathbf{q}}}(X_n) \right| \ll_{\epsilon, l, \eta, S, \mathbf{J}} \frac{n+1}{(\log N(\mathbf{n}))^3} + \frac{N(\mathbf{q}^n)^{-1/2+\epsilon}}{\log N(\mathbf{n})} + N(\mathbf{q}^n)^{2+c+\epsilon} N(\mathbf{n})^{-\inf(c, 1)+\epsilon}$$

for  $n \in \mathbb{N}_0$  and  $\mathbf{n} \in \mathcal{J}_{\mathbf{qa}_S}$  with  $N(\mathbf{n}) > N_{\epsilon, S, l}$ . Here the implied constant is independent of  $n$  and  $\mathbf{n}$ . Moreover,

$$(7.4) \quad \Omega_{\mathbf{n}}(\pi) \geq 0 \quad \text{for all } \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}) \text{ and } \mathbf{n} \in \mathcal{J}_{\mathbf{qa}_S}.$$

*Proof.* Given an integer  $M > 1$ , define  $\chi_j^M(x) = \sum_{n=0}^M \hat{\chi}_j(n) X_n(x)$  for  $x \in [-2, 2]$  with  $\hat{\chi}_j(n) = \int_{-2}^2 \chi_j(x) X_n(x) d\mu^{\text{ST}}(x)$  and set

$$\chi(\mathbf{x}) = \prod_{j=1}^r \chi_j(x_j), \quad \chi^M(\mathbf{x}) = \prod_{j=1}^r \chi_j^M(x_j)$$

for  $\mathbf{x} = \{x_j\}_{j=1}^r$  in the product space  $[-2, 2]^r$ . Let  $\mathbf{n} \in \mathcal{J}_{\mathbf{qa}_S}$ . By the triangle inequality, the left-hand side of (7.3) is no greater than the sum of the following three terms :

$$(7.5) \quad \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \omega_{\mathbf{n}}(\pi) X_n(\lambda_{\mathbf{q}}(\pi)) \{ \chi(\lambda_S(\pi)) - \chi^M(\lambda_S(\pi)) \} \right|,$$

$$(7.6) \quad \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \omega_{\mathbf{n}}(\pi) X_n(\lambda_{\mathbf{q}}(\pi)) \chi^M(\lambda_S(\pi)) - \mu_{\mathbf{q}, \eta_{\mathbf{q}}}(X_n) \mu_{S, \eta}(\chi^M) \right|,$$

$$(7.7) \quad | \{ \mu_{S, \eta}(\chi^M) - \mu_{S, \eta}(\chi) \} \mu_{\mathbf{q}, \eta_{\mathbf{q}}}(X_n) |,$$

where  $\lambda_S(\pi) = (\lambda_v(\pi))_{v \in S}$  and  $\mu_{S, \eta} = \otimes_{v \in S} \mu_{v, \eta_v}$ . Note  $\mu_{S, \eta}(\chi) = 1$ . We shall estimate these quantities. Since  $|\hat{\chi}_j(n)| \ll_{\chi_j} n^{-5}$  for any  $n > 0$  by integration-by-parts and by  $\max_{[-2, 2]} |X_n| \ll n+1$ , we have

$$|\chi_j^M(x)| \leq \sum_{n \leq M} |\hat{\chi}_j(n)| |X_n(x)| \ll_{\chi_j} \sum_{n \leq M} n^{-4} \leq \zeta(4)$$

and

$$\max_{x \in [-2, 2]} |\chi_j(x) - \chi_j^M(x)| \leq \sum_{n > M} |\hat{\chi}_j(n)| \max_{[-2, 2]} |X_n| \ll_{\chi_j} \sum_{n > M} n^{-4} \ll M^{-3}.$$

By these,

$$(7.8) \quad \max_{\mathbf{x} \in [-2, 2]^r} |\chi(\mathbf{x}) - \chi^M(\mathbf{x})| \leq \max_{\mathbf{x} \in [-2, 2]^r} \left( \sum_{j=1}^r \left| \prod_{h=1}^{j-1} \chi_h^M(x_h) \right| |\chi_j(x_j) - \chi_j^M(x_j)| \right) \ll_{S, \chi} M^{-3}.$$

From (7.1) for  $\mathfrak{a} = \mathfrak{o}$ , noting  $\mathbf{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S} \subset \mathcal{J}_{\mathfrak{o}}$ , we have the estimate  $|\sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \omega_{\mathbf{n}}(\pi) - 1| \ll_{\epsilon, l, \eta} (\log N(\mathbf{n}))^{-1} + N(\mathbf{n})^{-\inf(c, 1) + \epsilon}$ . Hence (7.5) is majorized by

$$\{\max_{[-2, 2]} |X_n|\} \left\{ \max_{\mathbf{x} \in [-2, 2]^r} |\chi(\mathbf{x}) - \chi^M(\mathbf{x})| \right\} \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \omega_{\mathbf{n}}(\pi) \ll_{\epsilon, l, \eta, S, \chi} (n+1) M^{-3} (1 + N(\mathbf{n})^{-\inf(c, 1) + \epsilon}).$$

By (7.8), the quantity (7.7) is majorized by  $\mu_{\mathfrak{q}, \eta_{\mathfrak{q}}}(X_n) M^{-3}$ , which amounts at most to  $(n+1) M^{-3}$ . Let us estimate (7.6). By expanding the product,  $\chi^M(\mathbf{x})$  is expressed as a sum of the terms  $\prod_{j=1}^r \hat{\chi}_j(n_j) \prod_{j=1}^r X_{n_j}(x_j)$  over all  $\mathbf{n} = (n_j)_{j=1}^M \in [0, M]^r$ . Hence by using (7.1), we can majorize (7.6) from above by

$$\begin{aligned} & \sum_{\mathbf{n}} \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \omega_{\mathbf{n}}(\pi) \prod_j X_{n_j}(\lambda_{v_j}(\pi)) - \mu_{S, \eta}(\prod_j X_{n_j}) \right| \\ & \ll_{\epsilon, l, \eta, S, \chi} \frac{N(\mathfrak{a}_S^M \mathfrak{q}^n)^{-1/2 + \epsilon}}{\log N(\mathbf{n})} + N(\mathfrak{a}_S^M \mathfrak{q}^n)^{2+c+\epsilon} N(\mathbf{n})^{-\inf(c, 1) + \epsilon}. \end{aligned}$$

Combining the estimations made so far, we have that the left-hand side of (7.3) is majorized by

$$(7.9) \quad (n+1) M^{-3} (1 + N(\mathbf{n})^{-\inf(c, 1) + \epsilon}) + \frac{N(\mathfrak{a}_S^M \mathfrak{q}^n)^{-1/2 + \epsilon}}{\log N(\mathbf{n})} + N(\mathfrak{a}_S^M \mathfrak{q}^n)^{2+c+\epsilon} N(\mathbf{n})^{-\inf(c, 1) + \epsilon}.$$

Now take

$$M = \left\lceil \frac{\epsilon}{2 + c + \epsilon} \frac{\log N(\mathbf{n})}{\log N(\mathfrak{a}_S)} \right\rceil.$$

Then  $N(\mathfrak{a}_S)^{M(2+c+\epsilon)} \leq N(\mathbf{n})^\epsilon$ , and also  $N(\mathfrak{a}_S)^{M(-1/2+\epsilon)} \leq 1$  evidently. By these, (7.9) is majorized by

$$\begin{aligned} & (n+1)(\log N(\mathbf{n}))^{-3} \log N(\mathfrak{a}_S)^3 (1 + N(\mathbf{n})^{-\inf(c, 1) + \epsilon}) + \frac{N(\mathfrak{q}^n)^{-1/2 + \epsilon}}{\log N(\mathbf{n})} + N(\mathfrak{q}^n)^{2+c+\epsilon} N(\mathbf{n})^{-\inf(c, 1) + 2\epsilon} \\ & \ll_{\epsilon, S} (n+1)(\log N(\mathbf{n}))^{-3} + \frac{N(\mathfrak{q}^n)^{-1/2 + \epsilon}}{\log N(\mathbf{n})} + N(\mathfrak{q}^n)^{2+c+\epsilon} N(\mathbf{n})^{-\inf(c, 1) + 2\epsilon}. \end{aligned}$$

□

**Lemma 39.** *Let  $I \subset [-2, 2]$  be an open interval disjoint from the set  $\{\lambda_{\mathfrak{q}}(\pi) \mid \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \Omega_{\mathbf{n}}(\pi) \neq 0\}$ . Then for any small  $\epsilon > 0$ , there exists a constant  $N_{\epsilon, l, \eta, S, \mathfrak{q}} > 0$  such that for any ideal  $\mathbf{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$  with  $N(\mathbf{n}) > N_{\epsilon, l, \eta, S, \mathfrak{q}}$ ,*

$$\mu_{\mathfrak{q}, \eta_{\mathfrak{q}}}(I) \ll_{\epsilon, l, \eta, S, \mathbf{J}} N(\mathfrak{q})^\epsilon (\log N(\mathbf{n}))^{-1 + \epsilon}$$

*holds with the implied constant independent of  $I$ ,  $\mathbf{n}$  and  $\mathfrak{q}$ .*

*Proof.* The proof of [4, Proposition 5.1 and Lemma 5.2] goes through as it is with a small modification. We reproduce the argument for convenience.

Let  $\Delta > 0$  be a parameter to be specified below and  $K$  a closed subinterval of  $I$  such that

$$(i) \quad \mu_{\mathfrak{q}, \eta_{\mathfrak{q}}}(I - K) \leq \Delta.$$

Depending on  $\Delta$  and  $K$ , we choose a  $C^\infty$ -function  $f$  on  $\mathbb{R}$  such that

- (ii)  $\text{supp}(f) \subset \bar{I}$ ,
- (iii)  $f(x) = 1$  if  $x \in K$  and  $0 \leq f(x) \leq 1$  for  $x \in \mathbb{R}$ ,
- (iv)  $|f^{(k)}(x)| \ll_k \Delta^{-k}$  for  $k \in \mathbb{N}_0$ .

Since  $I$  does not contain the relevant  $\lambda_{\mathbf{q}}(\pi)$ 's, from (ii) we have  $\Omega_{\mathbf{n}}(\pi)f(\lambda_{\mathbf{q}}(\pi)) = 0$  for all  $\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})$ . Using this, from (i), and (iii), we have the inequalities

$$(7.10) \quad \begin{aligned} \mu_{\mathbf{q}, \eta_{\mathbf{q}}}(I) &\leq \mu_{\mathbf{q}, \eta_{\mathbf{q}}}(K) + \Delta \leq \int_{-2}^2 f d\mu_{\mathbf{q}, \eta_{\mathbf{q}}} + \Delta \\ &\leq \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \Omega_{\mathbf{n}}(\pi) f(\lambda_{\mathbf{q}}(\pi)) - \int_{-2}^2 f d\mu_{\mathbf{q}, \eta_{\mathbf{q}}} \right| + \Delta. \end{aligned}$$

If we set  $f_M(x) = \sum_{n=0}^M \hat{f}(n) X_n(x)$ , then the first term of (7.10) is bounded by the sum of the following three terms

$$(7.11) \quad \left( \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} |\Omega_{\mathbf{n}}(\pi)| \right) \cdot \max_{[-2, 2]} |f - f_M|,$$

$$(7.12) \quad \int_{-2}^2 \max_{[-2, 2]} |f - f_M| d\mu_{\mathbf{q}, \eta_{\mathbf{q}}},$$

$$(7.13) \quad \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \Omega_{\mathbf{n}}(\pi) f_M(\lambda_{\mathbf{q}}(\pi)) - \int_{-2}^2 f_M d\mu_{\mathbf{q}, \eta_{\mathbf{q}}} \right|.$$

We remark that by the non-negativity of  $\Omega_{\mathbf{n}}(\pi)$ , the absolute value in (7.11) can be deleted. Then by the estimate  $|\hat{f}(n)| \ll_k n^{-k} \Delta^{-k}$  which follows from (iv) by integration by parts, and by  $\max_{[-2, 2]} |X_n| \ll n + 1$ , we have

$$\max_{[-2, 2]} |f - f_M| \leq \sum_{n > M} |\hat{f}(n)| \max_{[-2, 2]} |X_n| \ll_k \sum_{n > M} n^{-k} \Delta^{-k} n \ll M^{2-k} \Delta^{-k}$$

with  $k \geq 3$ . From (7.3) applied with  $n = 0$ , noting  $\mu_{\mathbf{q}, \eta_{\mathbf{q}}}(X_0) = 1$ , we have the estimate  $|\sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \Omega_{\mathbf{n}}(\pi) - 1| \ll_{\epsilon, l, \eta, S, \mathbf{J}} (\log N(\mathbf{n}))^{-1} + N(\mathbf{n})^{-\inf(c, 1) + \epsilon}$ . Hence the sum of (7.11) and (7.12) is majorized by

$$\Delta^{-k} M^{2-k} (1 + (\log N(\mathbf{n}))^{-1} + N(\mathbf{n})^{-\inf(c, 1) + \epsilon}) \ll \Delta^{-k} M^{2-k}$$

with the implied constant independent of  $\Delta$ ,  $M$ ,  $\mathbf{q}$  and  $\mathbf{n}$ . By (7.3) and by  $|\hat{f}(n)| \ll 1$ , the term (7.13) is majorized by

$$\begin{aligned} &\sum_{n=0}^M |\hat{f}(n)| \left| \sum_{\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})} \Omega_{\mathbf{n}}(\pi) X_n(\lambda_{\mathbf{q}}(\pi)) - \mu_{\mathbf{q}, \eta_{\mathbf{q}}}(X_n) \right| \\ &\ll_{\epsilon, l, \eta, S, \mathbf{J}} \sum_{n=0}^M \left( \frac{n+1}{(\log N(\mathbf{n}))^3} + \frac{N(\mathbf{q}^n)^{-1/2+\epsilon}}{\log N(\mathbf{n})} + N(\mathbf{q}^n)^{2+c+\epsilon} N(\mathbf{n})^{-\inf(c, 1) + \epsilon} \right) \\ &\ll_{\epsilon} \frac{M^2}{(\log N(\mathbf{n}))^3} + \frac{1}{\log N(\mathbf{n})} + N(\mathbf{q})^{c'M} N(\mathbf{n})^{-\inf(c, 1) + \epsilon}, \end{aligned}$$



where  $c' = 2 + c + \epsilon$ . Putting all relevant estimations together, we obtain

$$\mu_{\mathfrak{q}, \eta_{\mathfrak{q}}}(I) \ll_{k, \epsilon, l, \eta, S, \mathbf{J}} \Delta + \Delta^{-k} M^{2-k} + \frac{1}{\log N(\mathbf{n})} + \frac{M^2}{(\log N(\mathbf{n}))^3} + N(\mathfrak{q})^{c'M} N(\mathbf{n})^{-\inf(c, 1) + \epsilon}$$

with the implied constant independent of  $I$ ,  $\Delta$ ,  $M$ ,  $\mathfrak{q}$  and  $\mathbf{n}$ . By setting  $M = \left\lceil \frac{\inf(c, 1)}{2c'} \frac{\log N(\mathbf{n})}{\log N(\mathfrak{q})} \right\rceil$ , this yields the estimate

$$\mu_{\mathfrak{q}, \eta_{\mathfrak{q}}}(I) \ll_{k, \epsilon, l, \eta, S, \mathbf{J}} \Delta + \Delta^{-k} (\log N(\mathfrak{q}))^{k-2} (\log N(\mathbf{n}))^{2-k} + (\log N(\mathbf{n}))^{-1} + N(\mathbf{n})^{-c_2/2 + \epsilon}.$$

Let  $\epsilon > 0$  and we let  $\Delta$  vary so that it satisfies  $\Delta^{-k} (\log N(\mathbf{n}))^{2-k} \asymp_k (\log N(\mathbf{n}))^{-1+\epsilon}$ , or equivalently

$$\Delta \asymp_k (\log N(\mathbf{n}))^{-1+(3-\epsilon)/k}.$$

By taking  $k = \lceil 3/\epsilon \rceil + 1$ , we have  $(\log N(\mathbf{n}))^{-1+\epsilon/2} \ll_{\epsilon} \Delta \ll_{\epsilon} (\log N(\mathbf{n}))^{-1+\epsilon}$ . Hence,

$$\begin{aligned} \mu_{\mathfrak{q}, \eta_{\mathfrak{q}}}(I) &\ll_{\epsilon, l, \eta, S, \mathbf{J}} (\log N(\mathbf{n}))^{-1+\epsilon} + (\log N(\mathbf{n}))^{-1+\epsilon} (\log N(\mathfrak{q}))^{k-2} + (\log N(\mathbf{n}))^{-1} + N(\mathbf{n})^{-\inf(c, 1)/2 + \epsilon} \\ &\ll_{\epsilon} N(\mathfrak{q})^{\epsilon} (\log N(\mathbf{n}))^{-1+\epsilon}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 40.** *Given  $\epsilon > 0$ , there exists a positive number  $N_{\epsilon, l, \eta, S, \mathfrak{q}, \mathbf{J}}$  such that for any ideal  $\mathbf{n} \in \mathcal{I}_{\mathfrak{q} \mathfrak{a}_S}$  with  $N(\mathbf{n}) > N_{\epsilon, l, \eta, S, \mathfrak{q}, \mathbf{J}}$ , the inequality*

$$\#\{\lambda_{\mathfrak{q}}(\pi) \mid \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \Omega_{\mathbf{n}}(\pi) \neq 0\} \geq N(\mathfrak{q})^{-\epsilon} (\log N(\mathbf{n}))^{1-\epsilon}$$

*holds.*

*Proof.* [4, Lemma 5.3].  $\square$

7.1. Let  $\Gamma = \text{Aut}(\mathbb{C}/\mathbb{Q})$ . We let the group  $\Gamma$  act on the set of even weights by the rule  ${}^{\sigma}l = (l_{\sigma^{-1}ov})_{v \in \Sigma_{\infty}}$  for  $l = (l_v)_{v \in \Sigma_{\infty}}$  and  $\sigma \in \Gamma$ , regarding  $\Sigma_{\infty} = \text{Hom}(F, \mathbb{C})$ . Let  $\mathbb{Q}(l)$  be the fixed field of  $\text{Stab}_{\Gamma}(l)$ , which is a finite extension of  $\mathbb{Q}$ . From [8] (see [10] also), the Satake parameter  $A_v(\pi)$  belongs to  $\text{GL}(2, \bar{\mathbb{Q}})$  for any  $v \in \Sigma_{\text{fin}} - S(\mathbf{n})$  and the set  $\Pi_{\text{cus}}(l, \mathbf{n})$  has a natural action of the Galois group  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(l))$  in such a way that  $({}^{\sigma}\pi)_v \cong \pi_{\sigma^{-1}ov}$  for all  $v \in \Sigma_{\infty}$  and

$$(7.14) \quad q_v^{1/2} A_v({}^{\sigma}\pi) = \sigma(q_v^{1/2} A_v(\pi)) \quad \text{for all } v \in \Sigma_{\text{fin}} - S(\mathbf{n}).$$

The field of rationality of  $\pi \in \Pi_{\text{cus}}(l, \mathbf{n})$ , to be denoted by  $\mathbb{Q}(\pi)$ , is defined as the fixed field of the group

$$\{\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(l)) \mid {}^{\sigma}\pi = \pi\}.$$

From (7.14), by the strong multiplicity one theorem for  $\text{GL}(2)$ , we have

$$\mathbb{Q}(\pi) = \mathbb{Q}(l)(q_v^{1/2} \lambda_v(\pi) \mid v \in \Sigma_{\text{fin}} - S(\mathbf{n})).$$

**Proposition 41.** *Suppose  $l$  is a parallel weight, i.e., there exists  $k \in 2\mathbb{N}$  such that  $l_v = k$  for all  $v \in \Sigma_{\infty}$ . Let  $S$  be a finite subset of  $\Sigma_{\text{fin}} - S(\mathfrak{f}_{\eta})$  and  $\mathbf{J} = \{J_v\}_{v \in S}$  a family of closed subintervals of  $(-2, 2)$ . Given a sufficiently small  $\epsilon > 0$  and a prime ideal  $\mathfrak{q}$  prime to  $S \cup S(\mathfrak{f}_{\eta})$ , there exists a positive integer  $N_{\epsilon, l, \eta, S, \mathfrak{q}, \mathbf{J}}$  such that for any  $\mathbf{n} \in \mathcal{I}_{\mathfrak{q} \mathfrak{a}_S}$  with*

$N(\mathbf{n}) > N_{\epsilon, l, \eta, S, \mathbf{q}, \mathbf{J}}$ , there exists  $\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})$  such that  $\omega_{\mathbf{n}}(\pi) \neq 0$ ,  $\lambda_v(\pi) \in J_v$  for all  $v \in S$ , and

$$[\mathbb{Q}(\pi) : \mathbb{Q}] \geq \sqrt{\max \left\{ \frac{(1 - \epsilon) \log \log N(\mathbf{n})}{\log(16\sqrt{N(\mathbf{q})})} - 2\epsilon, 0 \right\}}.$$

*Proof.* By choosing  $C^\infty$ -functions  $\{\chi_v\}$  as above, we construct the weight function  $\Omega_{\mathbf{n}}(\pi)$ . We follow the proof of [4, Proposition 7.3]. Let  $d(\mathbf{n}, \Omega)$  denote the maximal degree of algebraic numbers  $\lambda_{\mathbf{q}}(\pi)$  ( $\pi \in \Pi_{\text{cus}}^*(l, \mathbf{n})$ ,  $\Omega_{\mathbf{n}}(\pi) \neq 0$ ). Then,

$$\begin{aligned} d(\mathbf{n}, \Omega) &\leq \max \{ [\mathbb{Q}(\pi) : \mathbb{Q}] \mid \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \Omega_{\mathbf{n}}(\pi) \neq 0 \} \\ &\leq \max \{ [\mathbb{Q}(\pi) : \mathbb{Q}] \mid \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \omega_{\mathbf{n}}(\pi) \neq 0, \lambda_v(\pi) \in J_v (\forall v \in S) \}. \end{aligned}$$

Let  $\mathcal{E}(M, d)$  denote the set of algebraic integers which, together with its conjugates, have the absolute values at most  $M$  and the absolute degrees at most  $d$ . From the parallel weight assumption, the Hecke eigenvalues  $N(\mathbf{q})^{1/2} \lambda_{\mathbf{q}}(\pi)$  are known to be algebraic integers ([8, Proposition 2.2]). Since  $\sigma(N(\mathbf{q})^{1/2} \lambda_{\mathbf{q}}(\pi)) = N(\mathbf{q})^{1/2} \lambda_{\mathbf{q}}(\sigma\pi)$  from (7.14), by the Ramanujan bound, we have  $N(\mathbf{q})^{1/2} \lambda_{\mathbf{q}}(\pi) \in \mathcal{E}(2N(\mathbf{q})^{1/2}, d(\mathbf{n}, \Omega))$ . Then the cardinality of the set  $\{N(\mathbf{q})^{1/2} \lambda_{\mathbf{q}}(\pi) \mid \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \Omega_{\mathbf{n}, \eta}(\pi) \neq 0\}$  is bounded from above by  $\#\mathcal{E}(2N(\mathbf{q})^{1/2}, d(\mathbf{n}, \Omega))$  which in turn is no greater than  $(16N(\mathbf{q})^{1/2})^{d(\mathbf{n}, \Omega)^2}$  by [4, Lemma 6.2]. Combining this with the lower bound provided by Lemma 40, we have

$$N(\mathbf{q})^{-\epsilon} (\log N(\mathbf{n}))^{1-\epsilon} \leq (16N(\mathbf{q})^{1/2})^{d(\mathbf{n}, \Omega)^2}.$$

By taking logarithm, we are done.  $\square$

**Remark :** The parallel weight assumption can be removed if the integrality of the Hecke eigenvalues  $q_v^{1/2} \lambda_v(\pi)$  for all  $v \in \Sigma_{\text{fin}} - S(\mathbf{f}_{\pi})$  is known in a broader generality.

**7.2. The proof of Theorem 3.** Theorem 1 means the numbers

$$\omega_{\mathbf{n}}(\pi) = \frac{C_l}{4D_F^{3/2} L_{\text{fin}}(1, \eta) \nu(\mathbf{n})} \frac{1}{N(\mathbf{n})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_{\pi}}(1, \pi; \text{Ad})}, \quad \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \mathbf{n} \in \mathcal{I}_{S \cup S(\mathbf{q}), \eta}^+$$

satisfy our first assumption (7.1). The second assumption (7.2) follows from [2]. Thus Theorem 3 is a corollary of Proposition 41 with this particular  $\{\omega_{\mathbf{n}}(\pi)\}$ .

**7.3. The proof of Theorem 4.** For any  $M > 1$ , let  $\mathcal{I}_{S \cup S(\mathbf{q}), \eta}^- [M]$  be the set of  $\mathbf{n} \in \mathcal{I}_{S \cup S(\mathbf{q}), \eta}^-$  such that  $\sum_{v \in S(\mathbf{n})} \frac{\log q_v}{q_v} \leq M$ . Theorem 1 means

$$\omega_{\mathbf{n}}(\pi) = \frac{C_l}{4D_F^{3/2} L_{\text{fin}}(1, \eta) \nu(\mathbf{n}) \log \sqrt{N(\mathbf{n})}} \frac{1}{N(\mathbf{n})} \frac{L(1/2, \pi) L'(1/2, \pi \otimes \eta)}{L^{S_{\pi}}(1, \pi; \text{Ad})}, \quad \pi \in \Pi_{\text{cus}}^*(l, \mathbf{n}), \mathbf{n} \in \mathcal{I}_{S \cup S(\mathbf{q}), \eta}^- [M]$$

satisfy our first assumption (7.1). By our non-negativity assumption (1.8), the second assumption (7.2) is also available. Thus Theorem 4 follows from Proposition 41.

**Remark :** In the parallel weight two case (i.e.,  $l_v = 2$  for all  $v \in \Sigma_{\infty}$ ) with totally imaginary condition on  $\eta$ , the assumption (1.8) follows from [16, Theorem 6.1] due to the non-negativity of the Neron-Tate height pairing. Similar results may be expected in the parallel higher weight case ([14]).

## 8. COMPUTATIONS OF LOCAL TERMS

Let  $\alpha = \otimes_{v \in S} \alpha_v$  be a decomposable element of  $\mathcal{A}_S$ . We examine the term  $\mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha)$  appearing in the formula (3.7), which is given in Lemma 15. Recall that the function  $\Psi_l^{(0)}(\mathbf{n}|\alpha, g)$  in adele points  $g = \{g_v\}$  is a product of functions  $\Psi_v(g_v)$  on local groups  $\text{GL}(2, F_v)$  such that  $\Psi_v(g_v) = \Psi_v^{(0)}(l_v; g_v)$  for  $v \in \Sigma_\infty$ ,  $\Psi_v(g_v) = \hat{\Psi}_v^{(0)}(\alpha_v; g_v)$  for  $v \in S$ , and  $\Psi_v(g_v) = \Phi_{\mathbf{n},v}^{(0)}(g_v)$  for  $v \in \Sigma_{\text{fin}} - S$  (see [7, §5]). From Lemma 15, by exchanging the order of integrals, we have the first equality of the formula

$$(8.1) \quad \begin{aligned} \mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n}|\alpha) &= \sum_{b \in F - \{0, -1\}} \int_{\mathbb{A}^\times} \hat{\Psi}_l^{(0)}(\mathbf{n}|\alpha, \delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_\eta \\ 0 & 1 \end{bmatrix}) \eta(tx_\eta^*) \log |t|_{\mathbb{A}} d^\times t \\ &= \sum_{b \in F - \{0, -1\}} \sum_{w \in \Sigma_F} \{ \prod_{v \in \Sigma_F - \{w\}} J_v(b) \} W_w(b), \end{aligned}$$

where

$$\begin{aligned} J_v(b) &= \int_{F_v^\times} \Psi_v(\delta_b \begin{bmatrix} t_v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_{\eta,v} \\ 0 & 1 \end{bmatrix}) \eta_v(t_v x_{\eta,v}^*) d^\times t_v, \\ W_w(b) &= \int_{F_w^\times} \Psi_w(\delta_b \begin{bmatrix} t_w & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_{\eta,w} \\ 0 & 1 \end{bmatrix}) \eta_w(t_w x_{\eta,w}^*) \log |t_w|_w d^\times t_w \end{aligned}$$

for  $b \in F_v - \{0, -1\}$ . The second equality of (8.1) is justified by  $\sum_b \sum_w \{ \prod_{v \neq w} |J_v(b)| \} |W_w(b)| < \infty$ , which results from the analysis to be made in 8.4. The integrals  $J_v(b)$  are studied and their explicit evaluations are obtained in [7, §10]. In what follows, we examine the integral  $W_w(b)$  separating cases  $w \in S$ ,  $w \in \Sigma_{\text{fin}} - S$  and  $w \in \Sigma_\infty$ .

8.1. Let  $v \in S$ . Then the integral  $W_v(b)$  depends on the test function  $\alpha_v \in \mathcal{A}_v$  and the character  $\eta_v$  of  $F_v^\times$ ; we write  $W_v^{\eta_v}(b; \alpha_v)$  in place of  $W_v(b)$  in this subsection. We have

$$W_v^{\eta_v}(b, \alpha_v) = \frac{1}{2\pi i} \int_{L_v(c)} \left\{ \int_{F_v^\times} \Psi_v^{(0)}(s_v; \delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t \right\} \alpha_v(s_v) d\mu_v(s_v).$$

**Lemma 42.** *Let  $v \in S$ . Let  $\alpha_v^{(m)}(s_v) = q_v^{ms_v/2} + q_v^{-ms_v/2}$  with  $m \in \mathbb{N}_0$ . Then, for any  $b \in F_v - \{0, -1\}$ ,*

$$W_v^{\eta_v}(b; \alpha_v^{(m)}) = \tilde{I}_v^+(m; b) + \eta_v(\varpi_v) \{ (\log q_v) I_v^+(m; \varpi_v^{-1}(b+1)) - \tilde{I}_v^+(m; \varpi_v^{-1}(b+1)) \}$$

with  $I_v^+(m; -)$  defined in [7, Lemma 10.2] and

$$\begin{aligned} \tilde{I}_v^+(m; b) &= \text{vol}(\mathfrak{o}_v^\times) (\log q_v) 2^{\delta(m=0)} \left( -q_v^{-m/2} \tilde{\delta}_m^{\eta_v}(b) \right. \\ &\quad \left. + \sum_{l=\sup(0, 1-\text{ord}_v(b))}^{m-1} \{ (m-l-1) q_v^{1-m/2} - (m-l+1) q_v^{-m/2} \} \tilde{\delta}_l^{\eta_v}(b) \right), \end{aligned}$$

where we set

$$\tilde{\delta}_n^{\eta_v}(b) = \delta(|b|_v < q_v^n) \eta_v(\varpi_v^n) \eta_v(b) (-n - \text{ord}_v(b))$$

for  $n \in \mathbb{N}$  and

$$\tilde{\delta}_0^{\eta_v}(b) = \delta(|b|_v < 1) \begin{cases} -2^{-1} \text{ord}_v(b)(\text{ord}_v(b) + 1), & (\eta_v(\varpi_v) = 1), \\ 4^{-1}(\eta_v(b) - 1) + 2^{-1} \text{ord}_v(b)\eta_v(b), & (\eta_v(\varpi_v) = -1). \end{cases}$$

When  $m = 0$ ,

$$W_v^{\eta_v}(b; \alpha_v^{(0)}) = -2\text{vol}(\mathfrak{o}_v^\times)(\log q_v)(\tilde{\delta}_0^{\eta_v}(b) + \eta_v(\varpi_v)\delta_0^{\eta_v}(\varpi_v^{-1}(b+1)) - \eta_v(\varpi_v)\tilde{\delta}_0^{\eta_v}(\varpi_v^{-1}(b+1)))$$

with  $\delta_0^{\eta_v}$  defined in [7, Lemma 10.2].

*Proof.* This is proved in a similar way to [7, Lemma 10.2]. We decompose the integral into the sum  $W_v(b; \alpha_v^{(m)}) = \tilde{I}_v^+(m; b) + \tilde{I}_v^-(m; b)$ , where  $\tilde{I}_v^+(m; b) = \int_{t \in F_v^\times, |t| \leq 1} \hat{\Phi}_{vm}(\delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t$  with  $\hat{\Phi}_{vm}(g_v)$  the integral computed in [7, Lemma 10.1]. We consider the case  $m > 0$ . By [7, Lemma 10.1],

$$\begin{aligned} \tilde{I}_v^+(m; b) &= \int_{|t| \leq 1, \sup(1, |t|_v^{-1}|b|_v) = q_v^m} (-q_v^{-m/2}) \eta_v(t) \log |t|_v d^\times t \\ &\quad + \sum_{l=0}^{m-1} \int_{|t| \leq 1, \sup(1, |t|_v^{-1}|b|_v) = q_v^l} \{(m-l-1)q_v^{1-m/2} - (m-l+1)q_v^{-m/2}\} \eta_v(t) \log |t|_v d^\times t. \end{aligned}$$

We have the following three equalities:

- If  $l = 0$  and  $\eta_v(\varpi_v) = 1$ ,

$$\int_{|t| \leq 1, \sup(1, |t|_v^{-1}|b|_v) = q_v^0} \eta_v(t) \log |t|_v d^\times t = \delta(|b|_v < 1) \text{vol}(\mathfrak{o}_v^\times) \log q_v \frac{-\text{ord}_v(b)(\text{ord}_v(b) + 1)}{2}.$$

- If  $l = 0$  and  $\eta_v(\varpi_v) = -1$ ,

$$\int_{|t| \leq 1, \sup(1, |t|_v^{-1}|b|_v) = q_v^0} \eta_v(t) \log |t|_v d^\times t = \delta(|b|_v < 1) \text{vol}(\mathfrak{o}_v^\times) \log q_v \left( \frac{\eta_v(b) - 1}{4} + \frac{\text{ord}_v(b)\eta_v(b)}{2} \right).$$

- If  $l > 0$ ,

$$\begin{aligned} \int_{|t| \leq 1, \sup(1, |t|_v^{-1}|b|_v) = q_v^l} \eta_v(t) \log |t|_v d^\times t &= \delta(|b|_v < q_v^l) \int_{|t|_v = q_v^{-l}|b|_v} \eta_v(t) \log |t|_v d^\times t \\ &= -\delta(|b|_v \leq q_v^l) \text{vol}(\mathfrak{o}_v^\times) (\log q_v) \eta_v(\varpi_v^l b) (l + \text{ord}_v(b)). \end{aligned}$$

Furthermore,  $\tilde{I}_v^-(m; b)$  is transformed into

$$\begin{aligned} \tilde{I}_v^-(m; b) &= \int_{|t|_v > 1} \hat{\Phi}_{vm}(\delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t \\ &= \int_{|y|_v < 1} \hat{\Phi}_{vm}(\delta_b \begin{bmatrix} \varpi_v^{-1} y^{-1} & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(\varpi_v^{-1} y^{-1}) \log |\varpi_v^{-1} y^{-1}|_v d^\times t \\ &= \eta_v(\varpi_v^{-1}) \int_{|y|_v \leq 1} \hat{\Phi}_{vm}(\delta_b \begin{bmatrix} \varpi_v^{-1} y^{-1} & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(y) (\log q_v - \log |y|_v) d^\times y \\ &= \eta_v(\varpi_v) \{(\log q_v) \tilde{I}_v^+(m; \varpi_v^{-1}(b+1)) - \tilde{I}_v^+(m; \varpi_v^{-1}(b+1))\}. \end{aligned}$$

From the results above, we have the lemma for  $m > 0$ . The case  $m = 0$  is similar.  $\square$

**Lemma 43.** For  $m \in \mathbb{N}$ ,

$$|W_v^{\eta_v}(b; \alpha_v^{(m)})| \ll (\log q_v) \delta(|b|_v \leq q_v^{m-1}) q_v^{1-m/2} m (2m + \text{ord}_v(b(b+1)))^2, \quad b \in F_v^\times - \{-1\}.$$

When  $m = 0$ ,

$$|W_v^{\eta_v}(b; \alpha_v^{(0)})| \ll (\log q_v) \delta(|b|_v \leq 1) (\text{ord}_v(b(b+1)) + 1)^2, \quad b \in F_v^\times - \{-1\}.$$

Here the implied constants independent of  $v$ ,  $m$  and  $b$ . Moreover, for  $n \in \mathbb{N}_0$ ,

$$|W_v^{\eta_v}(b; \alpha_{\mathfrak{p}_v^n})| \ll (\log q_v) q_v \delta(|b|_v \leq q_v^n) \{\text{ord}_v(b(b+1)) + 2n + 1\}^2, \quad b \in F_v^\times - \{-1\}$$

with the implied constant independent of  $v$ ,  $n$  and  $b$ .

*Proof.* Noting (5.1), by the first and second estimates in the lemma, the last estimate is given in the same way as in the proof of Proposition 31. We only prove the first estimate. Suppose  $m \geq 1$ . By [7, Lemma 10.2],  $I_v^+(m, \varpi_v^{-1}(b+1))$  is estimated as

$$|I_v^+(m, \varpi_v^{-1}(b+1))| \ll \delta(|b|_v \leq q_v^{m-1}) (m+1)^2 q_v^{1-m/2}.$$

Next we examine  $\tilde{I}_v^+(m; b)$ . From the definition of  $\tilde{\delta}_m^{\eta_v}$  (in Lemma 42), we have  $|\tilde{\delta}_0^{\eta_v}(b)| \leq \delta(|b|_v < 1) 2^{-1} (\text{ord}_v(b) + 1)^2$ . By using this,

$$\begin{aligned} & \sum_{l=\sup(0, 1-\text{ord}_v(b))}^{m-1} (m-l-1) q_v^{1-m/2} |\tilde{\delta}_l^{\eta_v}(b)| \\ & \leq \delta(m \geq 1, |b|_v \leq q_v^{m-2}) q_v^{1-m/2} \left\{ \sum_{l=1}^{m-1} (m-l-1) |\tilde{\delta}_l^{\eta_v}(b)| + (m-1) |\tilde{\delta}_0^{\eta_v}(b)| \right\} \\ & \leq \delta(m \geq 1, |b|_v \leq q_v^{m-2}) q_v^{1-m/2} \left\{ \sum_{l=1}^{m-1} (m-l-1)(l + \text{ord}_v(b)) + (m-1) |\tilde{\delta}_0^{\eta_v}(b)| \right\} \\ & = \delta(m \geq 1, |b|_v \leq q_v^{m-2}) q_v^{1-m/2} (m-1) \{6^{-1}(m-2)m + 2^{-1}(m-2)\text{ord}_v(b) + |\tilde{\delta}_0^{\eta_v}(b)|\} \\ & \ll \delta(m \geq 2, |b|_v \leq q_v^{m-2}) q_v^{1-m/2} m(m^2 + m\text{ord}_v(b) + (\text{ord}_v(b) + 1)^2) \\ & \ll \delta(m \geq 2, |b|_v \leq q_v^{m-2}) q_v^{1-m/2} m(m + \text{ord}_v(b))^2 \end{aligned}$$

Similarly,

$$\sum_{l=\sup(0, 1-\text{ord}_v(b))}^{m-1} (m-l+1) q_v^{-m/2} |\tilde{\delta}_l^{\eta_v}(b)| \ll \delta(m \geq 1, |b|_v \leq q_v^{m-2}) q_v^{-m/2} m(m + \text{ord}_v(b) + 1)^2.$$

Hence, we obtain

$$|\tilde{I}_v^+(m; b)| \ll (\log q_v) \delta(|b|_v \leq q_v^{m-1}) q_v^{1-m/2} m(m + \text{ord}_v(b))^2, \quad m \in \mathbb{N}, \quad b \in F_v^\times - \{-1\}.$$

Furthermore,

$$\begin{aligned} & |\tilde{I}_v^+(m; \varpi_v^{-1}(b+1))| \\ & \ll (\log q_v) \delta(|b+1|_v \leq q_v^{m-1}) q_v^{1-m/2} m(m + \text{ord}_v(b+1))^2, \quad m \in \mathbb{N}, \quad b \in F_v^\times - \{-1\}. \end{aligned}$$

As a consequence, we have the lemma.  $\square$

8.2. Let  $v \in \Sigma_{\text{fin}} - S$ . There are three cases to be considered:  $v \in \Sigma_{\text{fin}} - S(\mathfrak{nf}_\eta)$ ,  $v \in S(\mathfrak{n})$  and  $v \in S(\mathfrak{f}_\eta)$ .

**Lemma 44.** Let  $v \in \Sigma_{\text{fin}} - (S \cup S(\mathfrak{nf}_\eta))$ . For  $b \in F_v^\times - \{-1\}$ , we have

$$W_v^{\eta_v}(b) = \int_{F_v^\times} \Phi_{v,0}^{(0)}(\delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t = \text{vol}(\mathfrak{o}_v^\times) (\log q_v) \tilde{\Lambda}_v^{\eta_v}(b),$$

where

$$\tilde{\Lambda}_v^{\eta_v}(b) = \delta(|b|_v \leq 1) \begin{cases} \tilde{\delta}_0^{\eta_v}(b), & (|b|_v < 1), \\ -\tilde{\delta}_0^{\eta_v}(b+1), & (|b+1|_v < 1), \\ 0, & (|b|_v = |b+1|_v = 1). \end{cases}$$

In particular,  $|W_v^{\eta_v}(b)| \ll (\log q_v) \delta(|b(b+1)|_v < 1) (\text{ord}_v(b(b+1)) + 1)^2$ ,  $b \in F_v^\times - \{-1\}$ .

*Proof.* It follows immediately from the following computation:

$$\begin{aligned} \int_{F_v^\times} \Phi_{v,0}^{(0)}(\delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t &= \int_{|b|_v \leq |t|_v < 1} \eta_v(t) \log |t|_v d^\times t + \int_{|b+1|_v^{-1} \geq |t|_v > 1} \eta_v(t) \log |t|_v d^\times t \\ &= \text{vol}(\mathfrak{o}_v^\times) (\log q_v) \{ \tilde{\delta}_0^{\eta_v}(b) - \tilde{\delta}_0^{\eta_v}(b+1) \}. \end{aligned}$$

□

**Lemma 45.** Let  $v \in S(\mathfrak{n})$ . If  $\eta_v(\varpi_v) = 1$ , we have

$$\begin{aligned} W_v^{\eta_v}(b) &= \int_{F_v^\times} \Phi_{v,\mathfrak{n}}^{(0)}(\delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t \\ &= \text{vol}(\mathfrak{o}_v^\times) (-\log q_v) \delta(b \in \mathfrak{no}_v) 2^{-1} (\text{ord}_v(b) + \text{ord}_v(\mathfrak{n})) (\text{ord}_v(b) - \text{ord}_v(\mathfrak{n}) + 1). \end{aligned}$$

If  $\eta_v(\varpi_v) = -1$ , then

$$\begin{aligned} &W_v^{\eta_v}(b) \\ &= \text{vol}(\mathfrak{o}_v^\times) (-\log q_v) \delta(b \in \mathfrak{no}_v) [2^{-1} \{ \text{ord}_v(\mathfrak{n}) \eta_v(\varpi_v^{\text{ord}_v(\mathfrak{n})}) + \text{ord}_v(b) \eta_v(b) \} + 4^{-1} \{ \eta_v(b) - \eta_v(\varpi_v^{\text{ord}_v(\mathfrak{n})}) \}]. \end{aligned}$$

In particular,

$$|W_v^{\eta_v}(b)| \leq \delta(b \in \mathfrak{no}_v) (\log q_v) (\text{ord}_v(b) + \text{ord}_v(\mathfrak{n}) + 1)^2, \quad b \in F_v^\times - \{-1\}.$$

*Proof.* It follows immediately from the following computation:

$$\begin{aligned} \int_{F_v^\times} \Phi_{v,\mathfrak{n}}^{(0)}(\delta_b \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) \eta_v(t) \log |t|_v d^\times t &= \int_{|b|_v \leq |t|_v < 1} \delta(t \in \mathfrak{no}_v) \eta_v(t) \log |t|_v d^\times t \\ &= \delta(b \in \mathfrak{no}_v) \sum_{n=\text{ord}_v(\mathfrak{n})}^{\text{ord}_v(b)} \int_{\mathfrak{o}_v^\times} \eta_v(\varpi_v^n u) \log |\varpi_v^n u|_v d^\times u = \delta(b \in \mathfrak{no}_v) \text{vol}(\mathfrak{o}_v^\times) (-\log q_v) \sum_{n=\text{ord}_v(\mathfrak{n})}^{\text{ord}_v(b)} \eta_v(\varpi_v^n) n. \end{aligned}$$

□

**Lemma 46.** Let  $v \in S(\mathfrak{f}_\eta)$  and put  $f = f(\eta_v) \in \mathbb{N}$ . For  $b \in F_v^\times - \{-1\}$ ,

$$\begin{aligned} W_v^{\eta_v}(b) &= \delta(b \in \mathfrak{p}_v^{-f}) \eta_v(-1) (1 - q_v^{-1})^{-1} q_v^{-f-d_v/2} (\log q_v) \times [-f + \\ &\quad \eta_v(b(b+1)) \{ \delta(b \in \mathfrak{p}_v) (-f - \text{ord}_v(b)) + \delta(b \in \mathfrak{o}_v^\times) (-f + \text{ord}_v(b+1)) + \delta(b \notin \mathfrak{o}_v) (-f) q_v^{\text{ord}_v(b)} \}]. \end{aligned}$$

In particular,

$$|W_v^{\eta_v}(b)| \leq 6(\log q_v)q_v^{-f}\delta(|b|_v \leq q_v^f)\{f + \delta(|b|_v \leq 1)\text{ord}_v(b(b+1))\}, \quad b \in F_v^\times - \{-1\}.$$

*Proof.* We have the expression  $W_v^{\eta_v}(b) = \delta(b \in \mathfrak{p}_v^{-f})(W_{v,1}^{\eta_v}(b) + W_{v,2}^{\eta_v}(b))$  with

$$W_{v,1}^{\eta_v}(b) = \int_{\substack{-t \in \varpi_v^f U_v(f) \\ |t|_v |b+1|_v \leq 1}} \eta_v(t\varpi_v^{-f}) \log |t|_v d^\times t = \eta_v(-1)(-f \log q_v)q_v^{-f-d_v/2}(1 - q_v^{-1})^{-1}$$

and

$$W_{v,2}^{\eta_v}(b) = \int_{\substack{-t \in F_v^\times - \varpi_v^f U_v(f) \\ |1+t\varpi_v^{-f}|_v |b+t\varpi_v^{-f}(b+1)|_v \leq |t|_v}} \eta_v(t\varpi_v^{-f}) \log |t|_v d^\times t.$$

The integration domain of  $W_{v,2}^{\eta_v}(b)$  is a disjoint union of the sets  $D_l(b)$  ( $l \in \mathbb{Z}$ ) defined in [7, 10.2]. By [7, Lemmas 10.6, 10.7 and 10.8],

$$\begin{aligned} W_{v,2}^{\eta_v}(b) &= \sum_{l \in \mathbb{Z}} (-l \log q_v) \int_{D_l(b)} \eta_v(t\varpi_v^{-f}) d^\times t \\ &= \delta(|b|_v < 1 = |b+1|_v) \{(-f + \text{ord}_v(b+1) - \text{ord}_v(b)) \log q_v\} \eta_v\left(\frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1} q_v^{-f-d_v/2} \\ &\quad + \delta(|b|_v = |b+1|_v \geq 1) (-f \log q_v) \eta_v\left(\frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1} q_v^{-f+\text{ord}_v(b)-d_v/2} \\ &\quad + \delta(|b+1|_v < 1 = |b|_v) \{(-f + \text{ord}_v(b+1) - \text{ord}_v(b)) \log q_v\} \eta_v\left(\frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1} q_v^{-f-d_v/2} \\ &= \eta_v\left(\frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1} q_v^{-f-d_v/2} (\log q_v) \{ \delta(|b|_v < 1 = |b+1|_v) (-f - \text{ord}_v(b)) \\ &\quad + \delta(|b|_v = |b+1|_v \geq 1) (-f) q_v^{\text{ord}_v(b)} + \delta(|b+1|_v < 1 = |b|_v) (-f + \text{ord}_v(b+1)) \} \\ &= \eta_v\left(\frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1} q_v^{-f-d_v/2} (\log q_v) \{ \delta(b \in \mathfrak{p}_v) (-f - \text{ord}_v(b)) \\ &\quad + \delta(b \in \mathfrak{o}_v^\times) (-f + \text{ord}_v(b+1)) + \delta(b \notin \mathfrak{o}_v) (-f) q_v^{\text{ord}_v(b)} \}. \end{aligned}$$

This completes the proof.  $\square$

8.3. Let  $v \in \Sigma_\infty$  and fix an identification  $F_v \cong \mathbb{R}$ . In this paragraph, we abbreviate  $l_v$  to  $l$  omitting the subscript  $v$ . Let  $\varepsilon : \mathbb{R}^\times \rightarrow \{\pm 1\}$  be a character; thus  $\varepsilon$  is the sign character or the trivial one. From the proof of [7, Lemma 10.12], we have

$$\begin{aligned} W_v^\varepsilon(b) &= \int_{\mathbb{R}^\times} \left( \frac{1+it}{\sqrt{t^2+1}} \right)^l \{1 + i(bt^{-1} + t(b+1))\}^{-l/2} \varepsilon(t) \log |t|_v d^\times t \\ &= \int_{\mathbb{R}^\times} (1-it)^{-l/2} (1+b+t^{-1}bi)^{-l/2} \varepsilon(t) \log |t|_v d^\times t \\ &= W_+(b) + \varepsilon(-1) \overline{W_+(b)}, \end{aligned}$$

where we set

$$W_+(b) = i^{l/2} (1+b)^{-l/2} \int_0^\infty (t+i)^{-l/2} \left( t + \frac{bi}{b+1} \right)^{-l/2} t^{l/2-1} \log t \, dt.$$



Here is an explicit formula of  $W_+(b)$ .

**Lemma 47.** *Suppose  $l \geq 4$ . Then, for  $b \in \mathbb{R}^\times - \{-1\}$ , we have*

$$W_+(b) = -\pi i J_+(l; b) - A(b) - i B(b),$$

where

$$\begin{aligned} A(b) &= \sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k} \binom{l/2-1}{k} \left\{ \frac{b^k}{2} (\log |\frac{b}{b+1}|)^2 - \frac{\theta(b)^2}{2} b^k - \frac{9\pi^2}{8} (-1)^{k+l/2} (b+1)^k \right\} \\ &\quad + \sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k} \sum_{j=1}^{l/2-k-1} \binom{l/2-1}{k+j} \frac{(-1)^j}{j} \left( \sum_{m=1}^{j-1} \frac{1}{m} \{b^k + (-1)^{k+l/2} (b+1)^k\} - b^k \log |\frac{b}{b+1}| \right), \\ B(b) &= \sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k} \binom{l/2-1}{k} b^k \log |\frac{b}{b+1}| \theta(b) \\ &\quad - \sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k} \sum_{j=1}^{l/2-k-1} \binom{l/2-1}{k+j} \frac{(-1)^j}{j} \left\{ \frac{3\pi}{2} (-1)^{k+l/2} (b+1)^k + b^k \theta(b) \right\}, \end{aligned}$$

$\theta(b) = \pi/2$  if  $b(b+1) < 0$ ,  $\theta(b) = 3\pi/2$  if  $b(b+1) > 0$  and  $J_+(l; b)$  is the function defined in [7, Lemma 10.13].

*Proof.* For  $b \in \mathbb{R}^\times - \{-1\}$ , put  $g(z) = i^{l/2} (1+b)^{-l/2} (z+i)^{-l/2} \left(z + \frac{bi}{b+1}\right)^{-l/2} z^{l/2-1} (\log z)^2$ , where  $\log z = \log |z| + i \arg(z)$  with  $\arg(z) \in [0, 2\pi)$ . Then,  $g(z)$  is holomorphic on  $\mathbb{C} - (\mathbb{R}_{\geq 0} \cup \{-i, \frac{-bi}{b+1}\})$ . We note  $\frac{-bi}{b+1} \in i\mathbb{R} - \{0, -i\}$ . By Cauchy's integral theorem, we have

$$2\pi i \{\text{Res}_{z=-i} + \text{Res}_{z=\frac{-bi}{b+1}}\} g(z) = \int_{\epsilon}^R g(t) dt + \oint_{|z|=R} g(z) dz - \int_{\epsilon}^R g(te^{2\pi i}) - \oint_{|z|=\epsilon} g(z) dz.$$

with  $R$  sufficiently large and  $\epsilon > 0$  sufficiently small. By  $\lim_{R \rightarrow \infty} \oint_{|z|=R} g(z) dz = 0$ ,  $\lim_{\epsilon \rightarrow +0} \oint_{|z|=\epsilon} g(z) dz = 0$  and  $(\log t + 2\pi i)^2 = (\log t)^2 + 4\pi i \log t - 4\pi^2$ , we also have

$$2\pi i \{\text{Res}_{z=-i} + \text{Res}_{z=\frac{-bi}{b+1}}\} g(z) = -4\pi i W_+(b) + 4\pi^2 J_+(l; b).$$

Hence, we obtain

$$W_+(b) = -\frac{1}{2} \{\text{Res}_{z=-i} + \text{Res}_{z=\frac{-bi}{b+1}}\} g(z) - \pi i J_+(l; b).$$

Furthermore, a direct computation gives us

$$\begin{aligned} \text{Res}_{z=-i} g(z) &= \sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k} (-1)^{k+l/2} (b+1)^k \\ &\quad \times \left\{ \binom{l/2-1}{k} \frac{-9\pi^2}{4} + 2 \sum_{j=1}^{l/2-k-1} \binom{l/2-1}{k+j} \frac{(-1)^j}{j} \left( \sum_{m=1}^{j-1} \frac{1}{m} - \frac{3\pi}{2} i \right) \right\} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Res}_{z=\frac{ib}{b+1}} g(z) &= \sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k} b^k \times \left\{ \binom{l/2-1}{k} \left( \log \left| \frac{b}{b+1} \right| + \theta(b)i \right)^2 \right. \\ &\quad \left. + 2 \sum_{j=1}^{l/2-k-1} \binom{l/2-1}{k+j} \frac{(-1)^j}{j} \left( \sum_{m=1}^{j-1} \frac{1}{m} - \log \left| \frac{b}{b+1} \right| - \theta(b)i \right) \right\}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 48.** *Suppose  $l > 4$ . For any  $\epsilon > 0$ , we have*

$$|b(b+1)|^\epsilon |W_v^\epsilon(b)| \ll_{\epsilon, l} (1 + |b|)^{-l/2+2\epsilon}, \quad b \in \mathbb{R} - \{0, -1\}.$$

*Proof.* From Lemmas 29 and 47, for any  $\epsilon > 0$ ,  $|b(b+1)|^\epsilon W_+(b)$  is locally bounded around the points  $b = 0, -1$ . For  $b$  away from the set  $\{0, -1\}$ , we have

$$|W_+(b)| \leq |2b(b+1)|^{-l/4} \int_0^\infty t^{l/4} (t^2 + 1)^{-l/4} |\log t| \frac{dt}{t}$$

by  $t^2(b+1)^2 + b^2 \geq 2|t||b(b+1)|$ . Since  $l > 4$ , the last integral is convergent; hence the above inequality gives us  $|b(b+1)|^\epsilon |W_+(b)| \ll_{\epsilon, l} (1 + |b|)^{-l/2+2\epsilon}$  for large  $|b|$ .  $\square$

**8.4. The proof of Proposition 33.** We start from the formula (8.1) taking  $\alpha$  to be  $\alpha_a$  defined by (1.5). If we set

$$(8.2) \quad \mathbb{W}(T) = \sum_{b \in F - \{0, -1\}} \sum_{w \in T} \left\{ \prod_{v \in \Sigma_F - \{w\}} J_v(b) \right\} W_w(b)$$

for any subset  $T \subset \Sigma_F$ , then (8.1) can be written in the form

$$\mathbb{W}_{\text{hyp}}^\eta(l, \mathbf{n} | \alpha_a) = \mathbb{W}(\Sigma_\infty) + \mathbb{W}(S(\mathbf{a})) + \mathbb{W}(S(\mathbf{n})) + \mathbb{W}(S(\mathbf{f}_\eta)) + \mathbb{W}(\Sigma_{\text{fin}} - S(\mathbf{a}\mathbf{f}_\eta)).$$

We shall estimate each term in the right-hand side of this equality explicating the dependence on  $\mathbf{n}$  and  $\mathbf{a} = \prod_{v \in S(\mathbf{a})} \mathfrak{p}_v^{n_v}$ . Set  $c = (l/2 - 1)/d_F$ . For convenience, we collect here all the estimates used below (other than these, we also need Lemma 48). Let  $w_1 \in S(\mathbf{a})$ ,  $w_2 \in S(\mathbf{n})$ ,  $w_3 \in S(\mathbf{f}_\eta)$ ,  $w_4 \in \Sigma_{\text{fin}} - S(\mathbf{a}\mathbf{f}_\eta)$ , and  $w_5 \in \Sigma_\infty$ . Let  $\epsilon > 0$  be a small number. Then,

$$(8.3) \quad |J_{w_1}(b)| \ll \delta(b \in \mathbf{a}^{-1} \mathfrak{o}_{w_1}) q_{w_1} \{1 + \Lambda_{w_1}(b)\}, \quad |J_{w_2}(b)| \leq \delta(b \in \mathbf{n} \mathfrak{o}_{w_2}),$$

$$(8.4) \quad |J_{w_3}(b)| \ll \delta(b \in \mathbf{f}_\eta^{-1} \mathfrak{o}_{w_3}), \quad |J_{w_4}(b)| \leq \delta(b \in \mathfrak{o}_{w_4}) \Lambda_{w_4}(b),$$

$$(8.5) \quad |b(b+1)|_{w_5}^\epsilon |J_{w_5}(b)| \ll_{\epsilon, l, w_5} (1 + |b|_{w_5})^{-l_{w_5}/2+2\epsilon}$$

(note the difference of  $\ll$  and  $\leq$ ), and

$$(8.6) \quad |W_{w_1}(b)| \ll (\log q_{w_1}) q_{w_1} \delta(b \in \mathbf{a}^{-1} \mathfrak{o}_{w_1}) \{2n_{w_1} + \operatorname{ord}_{w_1}(b(b+1)) + 1\}^2,$$

$$(8.7) \quad |W_{w_2}(b)| \ll (\log q_{w_2}) \delta(b \in \mathbf{n} \mathfrak{o}_{w_2}) \{\operatorname{ord}_{w_2}(b) + \operatorname{ord}_{w_2}(\mathbf{n}) + 1\}^2,$$

$$(8.8) \quad |W_{w_3}(b)| \ll (\log q_{w_3}) \delta(b \in \mathbf{f}_\eta^{-1} \mathfrak{o}_{w_3}) \{2f(\eta_{w_3}) + \operatorname{ord}_{w_3}(b(b+1)) + 1\},$$

$$(8.9) \quad |W_{w_4}(b)| \ll (\log q_{w_4}) \delta(|b(b+1)|_{w_4} < 1) \Lambda_{w_4}(b)^2$$

for  $b \in F^\times$ , where all the constants implied by the Vinogradov symbol are independent of the ideals  $\mathbf{n}$ ,  $\mathbf{a}$  and the places  $w_i$  ( $1 \leq i \leq 5$ ). Indeed, the second estimate in (8.3)

and the both estimates of (8.4) follow from [7, Lemmas 10.5, 10.4 and Corollary 10.11] immediately. The estimate (8.5) is from Lemma 29. The first estimate in (8.3) is obtained in the proof of Proposition 31. The estimate (8.6) follows from Lemma 43, (8.7) is from Lemma 45, (8.8) is from Lemma 46, and (8.9) is from Lemma 44.

In the remaining part of this section, all the constants implied by Vinogradov symbol are independent of  $\mathfrak{n}$  and  $\mathfrak{a}$  (but may depend on  $l$ ,  $\eta$  and a given small number  $\epsilon > 0$ ).

**Lemma 49.** *We have*

$$|\mathbb{W}(\Sigma_\infty)| \ll_{\epsilon, l, \eta} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-c+\epsilon}.$$

*Proof.* Similarly to the proof of Proposition 31, by Lemma 48, we have  $|\mathbb{W}(\Sigma_\infty)| \ll_{\epsilon, l, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_0^\eta(l, \mathfrak{n}, \mathfrak{f}_\eta \prod_{v \in I} \mathfrak{p}_v^{n_v})$ . Then, the desired estimate is given by Proposition 30.  $\square$

**Lemma 50.** *We have*

$$|\mathbb{W}(S(\mathfrak{a}))| \ll_{\epsilon, l, \eta} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-c+\epsilon}.$$

*Proof.* By the estimates recalled above, the range of  $b$  in the summation (8.2) with  $T = S(\mathfrak{a})$  can be restricted to  $\mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1} - \{0, -1\}$ . If  $b \in \mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1}$ , then  $b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2$  is an ideal of  $\mathfrak{o}$ . From this, noting that  $\eta$  is unramified over  $S(\mathfrak{a})$ , we have the equality  $\text{ord}_w(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2) = 2n_w + \text{ord}_w(b(b+1))$  for any  $w \in S(\mathfrak{a})$ ; by taking summation over  $w \in S(\mathfrak{a})$ ,

$$\sum_{w \in S(\mathfrak{a})} \{2n_w + \text{ord}_w(b(b+1)) + 1\} \log q_w \leq \log N(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2) + \log N(\mathfrak{a}) \ll_{\epsilon, \eta} |N(b(b+1))|^{\epsilon/2} N(\mathfrak{a})^\epsilon.$$

Using this, from (8.6), (8.3), and (8.4), we obtain

$$\begin{aligned} \mathbb{W}(S(\mathfrak{a})) &\ll \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1} - \{0, -1\}} \sum_{w_1 \in S(\mathfrak{a})} \left\{ \prod_{v \in \Sigma_F - \{w_1\}} |J_v(b)| \right\} (\log q_{w_1}) q_{w_1} \{\text{ord}_{w_1}(b(b+1)) + 2n_{w_1} + 1\}^2 \\ &\ll_{\epsilon, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a})^{1+2\epsilon} \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}} |N(b(b+1))|^\epsilon \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\ &\leq C^{\#S(\mathfrak{a})} N(\mathfrak{a})^{2\epsilon+1} \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{f}_\eta \prod_{v \in I} \mathfrak{p}_v^{n_v}), \end{aligned}$$

where  $C$  is the implied constant in the first estimate of (8.3) and (8.6). Noting  $C^{\#S(\mathfrak{a})} \ll_\epsilon N(\mathfrak{a})^\epsilon$ , we obtain the assertion by Proposition 30.  $\square$

**Lemma 51.** *We have*

$$|\mathbb{W}(S(\mathfrak{n}))| \ll_{\epsilon, l, \eta} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-c+\epsilon}.$$

*Proof.* From the estimations recalled above,

$$\begin{aligned} &|\mathbb{W}(S(\mathfrak{n}))| \\ &\ll_{\epsilon, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1} - \{0, -1\}} \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \sum_{w_2 \in S(\mathfrak{n})} |W_{w_2}(b)|, \end{aligned}$$

where  $C$  is the implied constant in the first estimate of (8.3). By (8.7),

$$\begin{aligned} \sum_{w_2 \in S(\mathfrak{n})} |W_w^{\eta w}(b)| &\ll \sum_{w_2 \in S(\mathfrak{n})} (\log q_{w_2})(\text{ord}_{w_2}(\mathfrak{n}) + \text{ord}_{w_2}(b) + 1)^2 \\ &\ll \sum_{w_2 \in S(\mathfrak{n})} \text{ord}_{w_2}(\mathfrak{n})^2 (\log q_{w_2}) + \sum_{w_2 \in S(\mathfrak{n})} (\log q_{w_2}) \Lambda_{w_2}(b)^2 \ll_{\epsilon} N(\mathfrak{n})^{\epsilon} \prod_{v \in S(\mathfrak{n})} \Lambda_v(b)^2 \end{aligned}$$

for  $b \in \mathfrak{n}\mathfrak{f}_{\eta}^{-1}\mathfrak{a}^{-1}$ . From this, we obtain

$$\begin{aligned} |\mathbb{W}(S(\mathfrak{n}))| &\ll_{\epsilon, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) N(\mathfrak{n})^{\epsilon} \sum_{b \in \mathfrak{n}\mathfrak{f}_{\eta}^{-1}\mathfrak{a}^{-1} - \{0, -1\}} \prod_{v \in \Sigma_{\infty}} |J_v(b)| \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{a}\mathfrak{f}_{\eta})} \Lambda_v(b)^2 \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\ &= C^{\#S(\mathfrak{a})} N(\mathfrak{a}) N(\mathfrak{n})^{\epsilon} \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_0^{\eta}(l, \mathfrak{n}, \mathfrak{f}_{\eta}) \prod_{v \in I} \mathfrak{p}_v^{n_v}. \end{aligned}$$

Then, the desired estimate is given by Proposition 30.  $\square$

**Lemma 52.** *We have*

$$\mathbb{W}(S(\mathfrak{f}_{\eta})) \ll_{\epsilon, l, \eta} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-c+\epsilon}.$$

*Proof.* By the same argument as in the proof of Lemma 50, we have

$$\begin{aligned} \sum_{w \in S(\mathfrak{f}_{\eta})} \{2f(\eta_w) + \text{ord}_w(b(b+1)) + 1\} \log q_w &\leq \log N(b(b+1)\mathfrak{a}^2\mathfrak{f}_{\eta}^2) + \log N(\mathfrak{f}_{\eta}) \\ &\ll_{\epsilon, \eta} |N(b(b+1))|^{\epsilon} N(\mathfrak{a})^{2\epsilon} \end{aligned}$$

for  $b \in \mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_{\eta}^{-1}$ . From the estimations recalled as above, we obtain

$$\begin{aligned} \mathbb{W}(S(\mathfrak{f}_{\eta})) &\leq \sum_{b \in \mathfrak{n}\mathfrak{f}_{\eta}^{-1}\mathfrak{a}^{-1}} \sum_{w_3 \in S(\mathfrak{f}_{\eta})} \left\{ \prod_{v \in \Sigma_F - \{w_3\}} |J_v(b)| \right\} (\log q_{w_3}) \{2f(\eta_{w_3}) + \text{ord}_{w_3}(b(b+1)) + 1\} \\ &\ll_{\epsilon, l, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) \sum_{b \in \mathfrak{n}\mathfrak{f}_{\eta}^{-1}\mathfrak{a}^{-1}} |N(b(b+1))|^{\epsilon} N(\mathfrak{a})^{2\epsilon} \prod_{v \in \Sigma_{\infty}} |J_v(b)| \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{a}\mathfrak{f}_{\eta})} \Lambda_v(b) \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\ &\ll_{\epsilon, l, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a})^{1+2\epsilon} \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_{\epsilon}^{\eta}(l, \mathfrak{n}, \mathfrak{f}_{\eta}) \prod_{v \in I} \mathfrak{p}_v^{n_v}. \end{aligned}$$

Then, the desired estimate is given by Proposition 30.  $\square$

**Lemma 53.** *We have*

$$\mathbb{W}(\Sigma_{\text{fin}} - S(\mathfrak{a}\mathfrak{f}_{\eta})) \ll_{\epsilon, l, \eta} N(\mathfrak{a})^{c+2+\epsilon} N(\mathfrak{n})^{-c+\epsilon}.$$

*Proof.* In the summation on the left hand side of (8.2) with  $T = \Sigma_{\text{fin}} - S(\mathfrak{a}\mathfrak{f}_{\eta})$ , the range of  $(b, w)$  is restricted to  $b \in \mathfrak{n}\mathfrak{f}_{\eta}^{-1}\mathfrak{a}^{-1}$  and  $w \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) \cap T$ , due to the estimations

recalled above. Thus,

$$\begin{aligned}
& \mathbb{W}(\Sigma_{\text{fin}} - S(\mathfrak{anf}_\eta)) \\
&= \sum_{b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a}^{-1} - \{0, -1\}} \sum_{w \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{anf}_\eta)} \left\{ \prod_{v \in \Sigma_F - \{w_4\}} |J_v(b)| \right\} |W_{w_4}(b)| \\
&\ll_{\epsilon, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) \sum_{b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a}^{-1} - \{0, -1\}} \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\
&\quad \times \sum_{w_4 \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{anf}_\eta)} \left\{ \prod_{\substack{v \in \Sigma_{\text{fin}} - S(\mathfrak{af}_\eta) \\ v \neq w_4}} \Lambda_v(b) \right\} (\log q_{w_4}) \Lambda_{w_4}(b)^2 \\
&\ll_{\epsilon, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) \sum_{b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a}^{-1} - \{0, -1\}} \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\
&\quad \times \left\{ \sum_{w_4 \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{anf}_\eta)} \log q_{w_4} \right\} \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{af}_\eta)} \Lambda_v(b)^2 \\
&\ll_{\epsilon, \eta} C^{\#S(\mathfrak{a})} N(\mathfrak{a}) \sum_{b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a}^{-1} - \{0, -1\}} \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\
&\quad \times \prod_{v \in \Sigma_{\text{fin}} - S(\mathfrak{af}_\eta)} \Lambda_v(b)^2 \times N(\mathfrak{a})^{2\epsilon} |N(b(b+1))|^\epsilon \\
&= C^{\#S(\mathfrak{a})} N(\mathfrak{a})^{1+2\epsilon} \sum_{I \subset S(\mathfrak{a})} \mathcal{J}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{f}_\eta \prod_{v \in I} \mathfrak{p}_v^{n_v}).
\end{aligned}$$

Here we note

$$\sum_{w_4 \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{anf}_\eta)} \log q_{w_4} \ll_{\epsilon, \eta} N(\mathfrak{a})^{2\epsilon} |N(b(b+1))|^\epsilon, \quad b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a}^{-1} - \{0, -1\}.$$

Indeed, if  $b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a}^{-1}$ , we have  $b(b+1)\mathfrak{f}_\eta^2 \mathfrak{a}^2 \subset \mathfrak{o}$  and  $S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{anf}_\eta) \subset S(b(b+1)\mathfrak{f}_\eta^2 \mathfrak{a}^2)$ . Hence,

$$\begin{aligned}
\sum_{w_4 \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{anf}_\eta)} \log q_{w_4} &\leq \sum_{w_4 \in S(b(b+1)\mathfrak{f}_\eta^2 \mathfrak{a}^2)} \log q_{w_4} \leq \log N(b(b+1)\mathfrak{f}_\eta^2 \mathfrak{a}^2) \\
&\ll_\epsilon |N(b(b+1))N(\mathfrak{f}_\eta)^2 N(\mathfrak{a})^2|^\epsilon, \quad b \in \mathfrak{nf}_\eta^{-1} \mathfrak{a} - \{0, -1\}.
\end{aligned}$$

Therefore, the assertion follows from Proposition 30 and from  $C^{\#S(\mathfrak{a})} \ll_\epsilon N(\mathfrak{a})^\epsilon$ .  $\square$

As a consequence, Proposition 33 follows from Lemmas 49, 50, 51, 52 and 53.

**8.5. Unipotent terms.** We compute the local terms for  $\tilde{\mathbb{W}}_\mathfrak{u}^\eta(l, \mathfrak{n}|\alpha)$  at a place  $v \in S$ . For  $\alpha_v \in \mathcal{A}_v$ , set

$$(8.10) \quad U_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{(1 - \eta_v(\varpi_v) q_v^{-(s+1)/2})(1 - q_v^{(s+1)/2})} \alpha(s) d\mu_v(s),$$

$$(8.11) \quad \tilde{U}_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\eta_v(\varpi_v) \log q_v}{(1 - \eta_v(\varpi_v) q_v^{-(s+1)/2})^2 (1 - q_v^{-(s+1)/2}) q_v^{s+1}} \alpha(s) d\mu_v(s)$$

with  $d\mu_v(s) = 2^{-1} \log q_v (q_v^{(1+s)/2} - q_v^{(1-s)/2}) ds$  and  $\sigma > 0$ . The integral  $U_v^{\eta_v}$  is already computed in [7, Proposition 11.1]. In the same way, we have the following lemma easily.

**Lemma 54.** *For any  $m \in \mathbb{N}_0$ , we have*

$$\tilde{U}_v^{\eta_v}(\alpha_v^{(m)}) = -\delta(m > 0) q_v^{-m/2} (\log q_v) \begin{cases} \left\{ \frac{q_v-1}{2} m (-1)^m - \frac{3q_v+1}{4} (-1)^m + \frac{1-q_v}{4} \right\}, & (\eta_v(\varpi_v) = -1), \\ \left\{ \frac{(m-1)(m-2)}{2} q_v - \frac{m(m+1)}{2} \right\}, & (\eta_v(\varpi_v) = +1). \end{cases}$$

## 9. APPENDIX: AN ESTIMATION OF A CERTAIN LATTICE SUM

Let  $d \geq 1$  be an integer. We fix  $l = (l_j)_{1 \leq j \leq d} \in \mathbb{R}^d$  such that  $l_d \geq \dots \geq l_1 \geq 4$ , and consider a positive function  $f(x)$  on  $\mathbb{R}^d$  defined by

$$f(x) = \prod_{j=1}^d (1 + |x_j|)^{-l_j/2}, \quad x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d.$$

Given a  $\mathbb{Z}$ -lattice  $\Lambda \subset \mathbb{R}^d$  (of full rank), we define

$$\theta(\Lambda) = \sum_{b \in \Lambda - \{0\}} f(b).$$

Viewing this as a function in  $\Lambda$ , we need to compare its asymptotic size with a certain power of  $D(\Lambda)$ , the Euclidean volume of a fundamental domain of  $\mathbb{R}^d/\Lambda$ . To state the main result of this section, we need another quantity  $r(\Lambda)$  given by

$$r(\Lambda) = \frac{1}{2} \min_{b \in \Lambda - \{0\}} \|b\|.$$

**Theorem 55.** *Let  $F$  be a totally real number field of degree  $d$ . Let  $\Lambda_0$  and  $\Lambda$  be fractional ideals such that  $\Lambda \subset \Lambda_0$ ; we regard them as a  $\mathbb{Z}$ -lattice in  $\mathbb{R}^d$  by the embedding  $F \rightarrow \mathbb{R}^{\text{Hom}(F, \mathbb{R})} = \mathbb{R}^d$ . Then,*

$$\theta(\Lambda) \ll \{1 + r(\Lambda_0)\}^{dl_d/2} D(\Lambda_0)^{-1} D(\Lambda)^{(1-l_1/2)/d}$$

with the implied constant independent of  $\Lambda$  and  $\Lambda_0$ .

The proof is given at the last part of the next subsection after several lemmas.

**9.1. The proof of Theorem 55.** Let  $d\mu(\omega)$  denote the Euclidean measure on the sphere  $\mathbb{S}^{d-1} = \{x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d \mid \sum_{j=1}^d x_j^2 = 1\}$ .

**Lemma 56.** *For any  $\lambda = (\lambda_j) \in \mathbb{C}^d$  such that  $\text{Re}(\lambda_j) < 1$ , we have*

$$I(\lambda) = \int_{\mathbb{S}^{d-1}} \prod_{j=1}^d |\omega_j|^{-\lambda_j} d\mu(\omega) = 2\Gamma\left(\sum_{j=1}^d \frac{1-\lambda_j}{2}\right)^{-1} \prod_{j=1}^d \Gamma\left(\frac{1-\lambda_j}{2}\right).$$

*Proof.* The formula is obtained by computing the integral

$$(9.1) \quad \int_{\mathbb{R}^d} \exp(-\epsilon \|x\|^2) \prod_{j=1}^d |x_j|^{-\lambda_j} dx$$

in two different ways, where  $\epsilon > 0$  and  $\operatorname{Re}(\lambda_j) < 1$  for the absolute convergence of the integral. By expressing (9.1) as an iterating integral, we compute it as

$$\prod_{j=1}^d \int_{\mathbb{R}} e^{-\epsilon x_j^2} |x_j|^{-\lambda_j} dx_j = \prod_{j=1}^d \epsilon^{(\lambda_j-1)/2} \Gamma\left(\frac{1-\lambda_j}{2}\right) = \epsilon^{(\sum_{j=1}^d \lambda_j - d)/2} \prod_{j=1}^d \Gamma\left(\frac{1-\lambda_j}{2}\right)$$

on one hand. On the other hand, by the polar decomposition, (9.1) becomes

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{S}^{d-1}} e^{-\epsilon \rho^2} \prod_{j=1}^d |\rho \omega_j|^{-\lambda_j} \rho^{d-1} d\rho d\mu(\omega) \\ &= \left( \int_{\mathbb{S}^{d-1}} \prod_{j=1}^d |\omega_j|^{-\lambda_j} d\mu(\omega) \right) \left( \int_0^\infty e^{-\epsilon \rho^2} \rho^{-\sum_{j=1}^d \lambda_j + d-1} d\rho \right) \\ &= I(\lambda) 2^{-1} \epsilon^{(\sum_{j=1}^d \lambda_j - d)/2} \Gamma\left(\sum_{j=1}^d \frac{1-\lambda_j}{2}\right). \end{aligned}$$

□

**Lemma 57.** For  $t = (t_j)_{1 \leq j \leq d} \in [1, \infty)^d$ , set

$$\varphi(t_1, \dots, t_d) = \int_{\mathbb{S}^{d-1}} f(t_1 \omega_1, \dots, t_d \omega_d) d\mu(\omega).$$

For  $t > 1$ , let  $\underline{t}$  denote the diagonal element  $(t_j)$  defined by  $t_j = t$  ( $1 \leq j \leq d$ ). Then,

$$\varphi(\underline{t}) = O(t^{1-d-l_1/2}), \quad t \in [1, \infty).$$

*Proof.* For  $\lambda = (\lambda_j) \in \mathbb{C}^d$  such that  $0 < \operatorname{Re}(\lambda_j) < 1$ , we compute the multiple Mellin-transform

$$\tilde{\varphi}(\lambda) = \int_0^\infty \cdots \int_0^\infty \varphi(t_1, \dots, t_d) \prod_{j=1}^d t_j^\lambda \frac{dt_j}{t_j}.$$

By Lemma 56, we compute this in the following manner.

$$\begin{aligned} \tilde{\varphi}(\lambda) &= \int_{\mathbb{S}^{d-1}} \left\{ \prod_{j=1}^d \int_0^\infty (1 + t_j |\omega_j|)^{-l_j/2} t_j^{\lambda_j-1} dt_j \right\} d\mu(\omega) \\ &= \int_{\mathbb{S}^{d-1}} \left\{ \prod_{j=1}^d |\omega_j|^{-\lambda_j} \int_0^\infty (1 + t_j)^{-l_j/2} t_j^{\lambda_j-1} dt_j \right\} d\mu(\omega) \\ &= \left( \int_{\mathbb{S}^{d-1}} \prod_{j=1}^d |\omega_j|^{-\lambda_j} d\mu(\omega) \right) \left( \prod_{j=1}^d \int_0^\infty (1 + t_j)^{-l_j/2} t_j^{\lambda_j-1} dt_j \right) \\ &= I(\lambda) \left\{ \prod_{j=1}^d \Gamma(l_j/2)^{-1} \Gamma(l_j/2 - \lambda_j) \Gamma(\lambda_j) \right\} \\ &= 2\Gamma\left(\sum_{j=1}^d \frac{1-\lambda_j}{2}\right)^{-1} \left\{ \prod_{j=1}^d \Gamma(l_j/2)^{-1} \Gamma((1 - \lambda_j)/2) \Gamma(l_j/2 - \lambda_j) \Gamma(\lambda_j) \right\}. \end{aligned}$$

By Stirling's formula, this is bounded by a constant multiple of  $P(\text{Im}\lambda) \exp(-\pi \sum_{j=1}^d |\text{Im}(\lambda_j)|)$  with some polynomial  $P(x_1, \dots, x_d)$  which can be taken uniformly with  $\text{Re}(\lambda)$  varied compactly. Thus, by a successive application of the Mellin inversion formula, we obtain

$$\varphi(\underline{t}) = \left( \frac{1}{2\pi i} \right)^d \int_{(\sigma_1)} \cdots \int_{(\sigma_d)} 2 \left\{ \prod_{j=1}^d \frac{\Gamma\left(\frac{1-\lambda_j}{2}\right) \Gamma\left(\frac{l_j}{2} - \lambda_j\right) \Gamma(\lambda_j)}{\Gamma(l_j/2)} \right\} \frac{t^{-\sum_{j=1}^d \lambda_j}}{\Gamma\left(\sum_{j=1}^d \frac{1-\lambda_j}{2}\right)} \prod_{j=1}^d d\lambda_j,$$

where the contour  $(\sigma_j) = \{\text{Re}(\lambda) = \sigma_j\}$  should be contained in the band  $0 < \text{Re}(\lambda_j) < 1$ . We shift the contours  $(\sigma_j)$  in some order far to the right. The residues arise when the moving contour  $(\sigma_j)$  passes the points in  $(1 + 2\mathbb{Z}_{\geq 0}) \cup (l_j/2 + \mathbb{Z}_{\geq 0})$ . Among those residues, the one with the smallest possible power of  $t^{-1}$  comes from the pole at  $\lambda_1 = l_1/2$ ,  $\lambda_j = 1$  ( $2 \leq j \leq d$ ) if  $l_2 > l_1$ , which we assume for simplicity in the rest of the proof of this lemma. (When  $l_2 = l_1$ , there are several terms giving the same power in  $t^{-1}$ .) The residue term is  $O(t^{-(d-1+l_2/2)})$ , by which the contribution from the remaining terms are majorized. This completes the proof.  $\square$

**Lemma 58.** (1)

$$(9.2) \quad f(x+y) \geq f(x)f(y), \quad x, y \in \mathbb{R}^d$$

(2)

$$\text{vol}(\mathbb{S}^{d-1}) (1+\rho)^{-dl_d/2} \leq \int_{\mathbb{S}^{d-1}} f(\rho\omega) d\mu(\omega) \ll (1+\rho)^{1-d-l_1/2}, \quad \rho > 0,$$

with the implied constant depending on  $l$  and  $d$ .

*Proof.* (1) is immediate from the inequality  $1 + |x_j + y_j| \leq (1 + |x_j|)(1 + |y_j|)$ . As for (2), we first note the inequality  $0 \leq |\omega_j| \leq 1$  for  $\omega \in \mathbb{S}^{d-1}$ . Using this, we have  $\prod_{j=1}^d (1 + |\rho\omega_j|) \leq (1 + \rho)^d$ . By this,

$$f(\rho\omega) \geq \left\{ \prod_{j=1}^d (1 + |\rho\omega_j|) \right\}^{-l_d/2} \geq (1 + \rho)^{-dl_d/2}.$$

Taking integral in  $\omega$ , we have the estimation from below as desired. The upper bound is provided by Lemma 57.  $\square$

We compare  $\theta(\Lambda)$  with the integral of  $f(x)$  on the ball  $B_\Lambda = \{x \in \mathbb{R}^d \mid \|x\| \leq r(\Lambda)\}$ . For convenience, we set  $I(D) = \int_D f(x) dx$  for any Borel set  $D$  in  $\mathbb{R}^d$ .

**Lemma 59.** Let  $\Lambda_0$  and  $\Lambda$  be  $\mathbb{Z}$ -lattices such that  $\Lambda \subset \Lambda_0$ . Then, we have the inequality

$$\theta(\Lambda) \leq I(B_{\Lambda_0})^{-1} I(\mathbb{R}^d - B_\Lambda)$$

*Proof.* The inequality (9.2) gives us

$$I(B_\Lambda) \theta(\Lambda) \leq \sum_{b \in \Lambda - \{0\}} \int_{B_\Lambda} f(b+x) dx.$$



Since  $\Lambda \subset \Lambda_0$ , we have  $B_{\Lambda_0} \subset B_\Lambda$ , from which  $I(B_{\Lambda_0}) \leq I(B_\Lambda)$  is obtained by the non-negativity of  $f(x)$ . Since  $(B_\Lambda + B_\Lambda) \cap \Lambda = \{0\}$ , the translated sets  $B_\Lambda + b$  ( $b \in \Lambda - \{0\}$ ) are mutually disjoint. From this remark,

$$\sum_{b \in \Lambda - \{0\}} \int_{B_\Lambda} f(b+x) dx \leq \int_{\mathbb{R}^d - B_\Lambda} f(x) dx = I(\mathbb{R}^d - B_\Lambda).$$

Putting altogether, we are done.  $\square$

**Lemma 60.** *Let  $\Lambda$  be a  $\mathbb{Z}$ -lattice.*

$$\begin{aligned} I(B_\Lambda) &\geq \text{vol}(\mathbb{S}^{d-1}) (1 + r(\Lambda))^{-dl_d/2} r(\Lambda)^d/d, \\ I(\mathbb{R}^d - B_\Lambda) &\leq r(\Lambda)^{1-l_1/2} \end{aligned}$$

with the implied constant independent of  $\Lambda$ .

*Proof.* By Lemma 58 (2),

$$\begin{aligned} I(B_\Lambda) &= \int_0^{r(\Lambda)} \int_{\mathbb{S}^{d-1}} f(\rho\omega) d\omega \rho^{d-1} d\rho \\ &\geq \text{vol}(\mathbb{S}^{d-1}) \int_0^{r(\Lambda)} (1 + \rho)^{-dl_d/2} \rho^{d-1} d\rho \\ &\geq \text{vol}(\mathbb{S}^{d-1}) (1 + r(\Lambda))^{-dl_d/2} \int_0^{r(\Lambda)} \rho^{d-1} d\rho = \text{vol}(\mathbb{S}^{d-1}) (1 + r(\Lambda))^{-dl_d/2} r(\Lambda)^d/d. \end{aligned}$$

In a similar way,

$$\begin{aligned} I(\mathbb{R}^d - B_\Lambda) &= \int_{r(\Lambda)}^\infty \int_{\mathbb{S}^{d-1}} f(\rho\omega) d\omega \rho^{d-1} d\rho \\ &\leq \int_{r(\Lambda)}^\infty (1 + \rho)^{1-d-l_1/2} \rho^{d-1} d\rho \leq \int_{r(\Lambda)}^\infty \rho^{-l_1/2} d\rho = (l_1/2 - 1)^{-1} r(\Lambda)^{1-l_1/2}. \end{aligned}$$

$\square$

**Lemma 61.** *Let  $F$  be a totally real number field of degree  $d$ . There exist constants  $C_d$  and  $C'_d$  such that  $C_d r(\Lambda)^d \leq D(\Lambda) \leq C'_d r(\Lambda)^d$  for any fractional ideal  $\Lambda$ .*

*Proof.* The first inequality follows from Minkowski's convex body theorem. The second inequality is proved as follows. For any  $b \in \Lambda - \{0\}$ , there exists an ideal  $\mathfrak{a} \subset \mathfrak{o}$  such that  $(b) = \mathfrak{a}\Lambda$ ; hence  $|N(b)| = N(\Lambda)N(\mathfrak{a}) \geq N(\Lambda)$ . Thus, by the arithmetic-geometric mean inequality,

$$D(\Lambda)^{1/d} = N(\Lambda)^{1/d} \leq \left\{ \prod_{j=1}^d |b_j|^2 \right\}^{1/(2d)} \leq \left\{ \sum_{j=1}^d |b_j|^2/d \right\}^{1/2} = d^{-1/2} \|b\|$$

Hence,  $D(\Lambda)^{1/d} \leq 2d^{-1/2} r(\Lambda)$ . This shows  $D(\Lambda) \leq C'_d r(\Lambda)^d$  with  $C'_d = (2d^{-1/2})^d$ .  $\square$

Theorem 55 follows from Lemmas 59, 60 and 61 immediately.

## ACKNOWLEDGEMENTS

The first author was supported by Grant-in-Aid for JSPS Fellows (25·668). The second author was supported by Grant-in-Aid for Scientific Research (C) 22540033.

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